New symmetries of GR ?

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based on *The Weyl BMS group and Einstein's equations* (arXiv:2104.05793)
with Laurent Freidel, Roberto Oliveri and Daniele Pranzetti

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Outline:

I - Noether theorem and e-m surface charges

II - Surface charges in GR and the BMS groups

III - The Weyl BMS group

I - Noether theorem
and e-m surface charges

Noether's theorem

If a Lagrangian has a continuous symmetry, then:

1. there exist a current which is conserved on shell,
2. the integral of the current defines a charge
   - conserved in time
   - canonical generator of the symmetry

Noether's theorem, in formulas: *(Throughout the talk, it will be convenient to work with Lagrangians as 4-forms)*

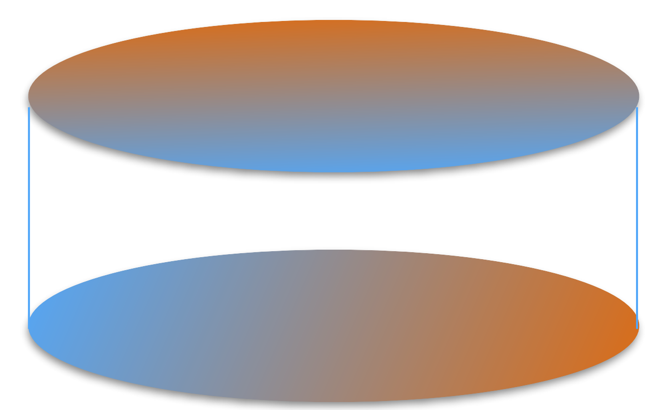
if $\delta_\epsilon \phi \;\Rightarrow\; \delta_\epsilon L = dY_\epsilon$ then $dj_\epsilon \approx 0$ $\quad j_\epsilon := \theta(\delta_\epsilon) - Y_\epsilon$

Proof:

$$\delta L = E + d\theta(\delta)$$

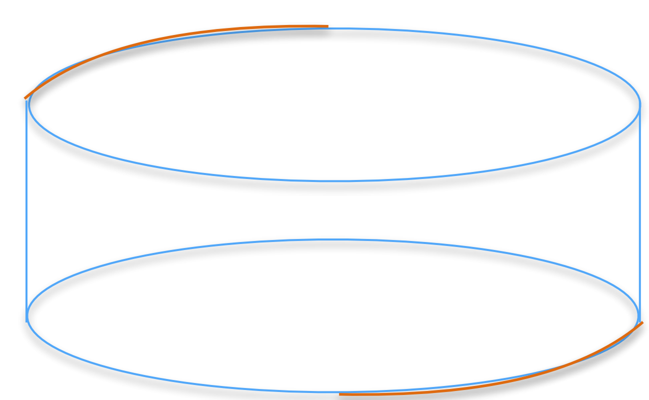
Typical examples:

- Poincaré invariance ⇨ conserved energy-momentum tensor
- ***Global*** U(1) invariance ⇨ conservation of electric charge



Less known examples:

- Diffeomorphisms ⇨ conservation laws for *surface* charges
  (in vacuum, with isometries)
- ***Local*** U(1) invariance ⇨ conservation laws for *surface* charges
  (in vacuum)



Surface charges: electromagnetism

Lagrangian: $L = -\frac{1}{4}F^2 = -\frac{1}{2}F \wedge \star F$ $\theta(\delta) = -\delta A \wedge \star F$

$$\delta L = -d\delta A \wedge \star F = -\delta A \wedge d \star F - d(\delta A \wedge \star F)$$

Surface charges: electromagnetism

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Surface charges: electromagnetism

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Gauge transformations: $$\delta_\lambda A = -d\lambda \qquad \delta_\lambda L = 0$$

Noether current: $$j_\lambda = \theta(\delta_\lambda) = d\lambda \wedge \star F = d(\lambda \star F) - \lambda d \star F \approx d(\lambda \star F)$$

Noether charge: $$Q = \int_\Sigma j$$

Surface charges: electromagnetism

Lagrangian: $L = -\frac{1}{4}F^2 = -\frac{1}{2}F \wedge \star F$ $\quad\quad$ $\theta(\delta) = -\delta A \wedge \star F$

Gauss law (the constraint associated with the gauge symmetry)

Gauge transformations: $\delta_\lambda A = -d\lambda$ $\quad\quad$ $\delta_\lambda L = 0$

Noether current: $j_\lambda = \theta(\delta_\lambda) = d\lambda \wedge \star F = d(\lambda \star F) - \lambda d \star F \approx d(\lambda \star F)$

Noether charge: $Q = \int_\Sigma j = \int_{\partial\Sigma} \lambda \star F$

Without boundaries, or with trivial gauge transformations at the boundaries, no charges for the e-m field

With boundaries and non-trivial gauge transformations we have access to infinitely many surface charges and their conservation laws

Surface charges: electromagnetism

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Without boundaries, or with trivial gauge transformations at the boundaries, no charges for the e-m field

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Coupling matter:

$$Q = \int_\Sigma j = \underbrace{\int_\Sigma \lambda J_\psi}_{\text{'hard'}} + \underbrace{\int_\Sigma d\lambda \star F}_{\text{'soft'}}$$

Global transformations: electric charge

Local transformations: balance law between the matter and e-m multipole moments

The case of null infinity

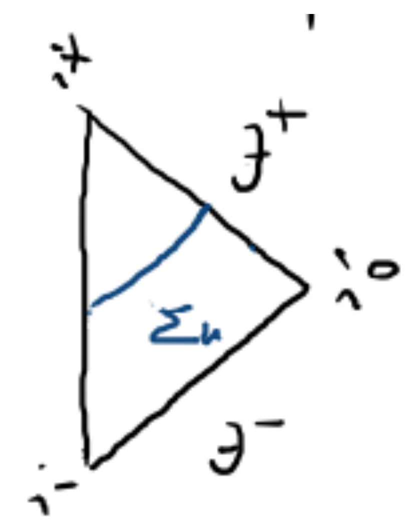
In which physical context are we interested in non-vanishing gauge transformations at the boundary?

An important example that has come to prominence in recent years (Bieri-Garfinkle '13, Strominger '14, …) concerns IR problems in scattering theory

Consider compactified Minkowski spacetime:

For a standard Cauchy hypersurface,
there is a natural choice of boundary conditions:

$$A_a \overset{i^0}{\mapsto} 0$$



For null Cauchy hypersurface intersecting future null infinity,
care is needed, as some choices would rule out electromagnetic radiation

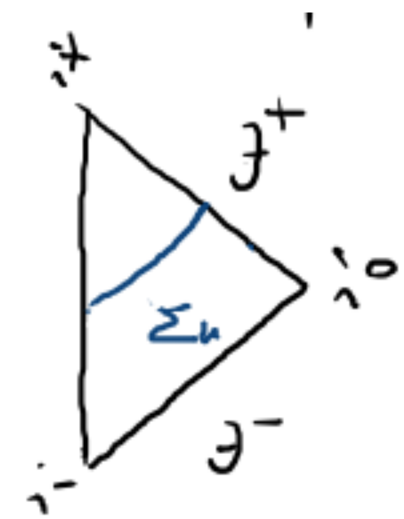
$$A_a \overset{\mathcal{I}}{\mapsto} 0 \quad \text{too fast means no waves}$$

Choosing appropriate b.c. at null infinity is delicate:
too strong may rule out physics, too weak may lead to divergences

Surface charges and electromagnetic memory

(Bieri-Garfinkle ’13, Strominger ’14, Ashtekar-Bonga ’17, Pastersky ’16)

Choosing the *weakest* possible conditions compatible with finite energy, one discovers that there is a **residual gauge freedom** at null infinity: $\lambda = \lambda(\theta, \phi)$



(It can be eliminated with stronger boundary conditions, but again, this may rule out interesting physics) $A_z \sim \mathcal{O}(1)\ , \quad A_r \sim \mathcal{O}\left(\frac{1}{r^2}\right)\ , \quad A_u \sim \mathcal{O}\left(\frac{1}{r}\right)$

Finding the right boundary and fall off conditions is a trial and error procedure, or in nicer words, a mix of art and science.

Surface charges offer a precious tool to discriminate: changing the boundary conditions means changing the asymptotic symmetry group

Consider now the integral of the Noether current on a portion Σ of null infinity:

$$Q = \int_\Sigma j = \underbrace{\int_\Sigma \lambda J_\psi}_{\text{‘hard’}} + \underbrace{\int_\Sigma d\lambda \star F}_{\text{‘soft’}}$$

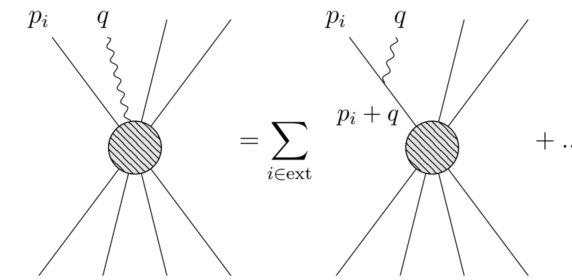
$$\lambda = \lambda(\theta, \phi) \qquad \Rightarrow \qquad \int du\delta\Omega\, d\lambda \wedge \star F = \int d\Omega \left(d\lambda \wedge \underbrace{\int du \star F}_{\text{‘e-m memory’}} \right)$$

- zero Fourier mode (⇨ soft)
- has the physical effect of a velocity kick to the particles in the detector

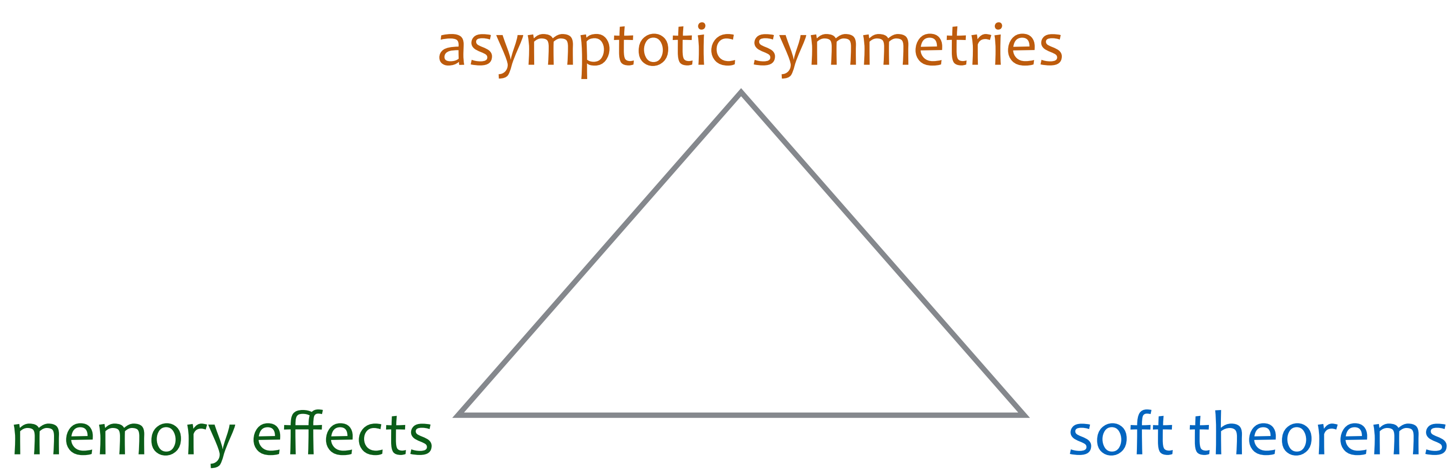
$$m\ddot{\vec{x}} = q\vec{E} \qquad \to \qquad \Delta\vec{v} = \frac{q}{m}\int_{-\infty}^{u} \vec{E}$$

Strominger's infrared triangle

The big surprise (Strominger et al) is that the Ward identities of the large gauge symmetry $\lambda = \lambda(\theta, \phi)$ reproduce Weinberg's soft theorem



A remarkable set of relationships between seemingly unrelated theoretical results (and communities) whose implications are just beginning to be explored



A similar story is unfolding in gravity, but there the identification of the `weakest possible boundary conditions' or equivalently 'largest possible symmetry group' is not over yet

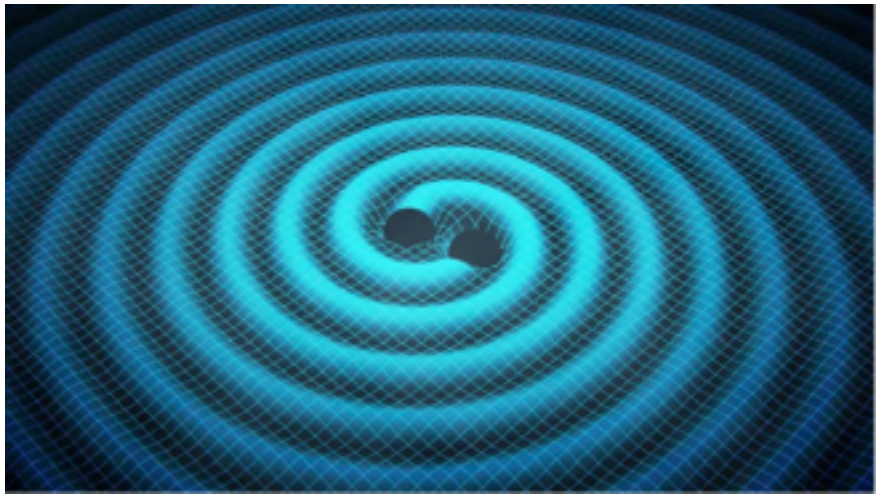
II - Surface charges in GR and the BMS group

Symmetries in general relativity

General relativity has no global symmetries

Only special solutions admit global symmetries, which are referred to as isometries, since they preserve the metric: e.g.

- the Poincaré symmetry of the Minkowski metric
- the spherical or axial symmetry of the Schwarzschild and Kerr black holes
- the spatial symmetries (e.g. homogeneity, isotropy) of cosmological metrics

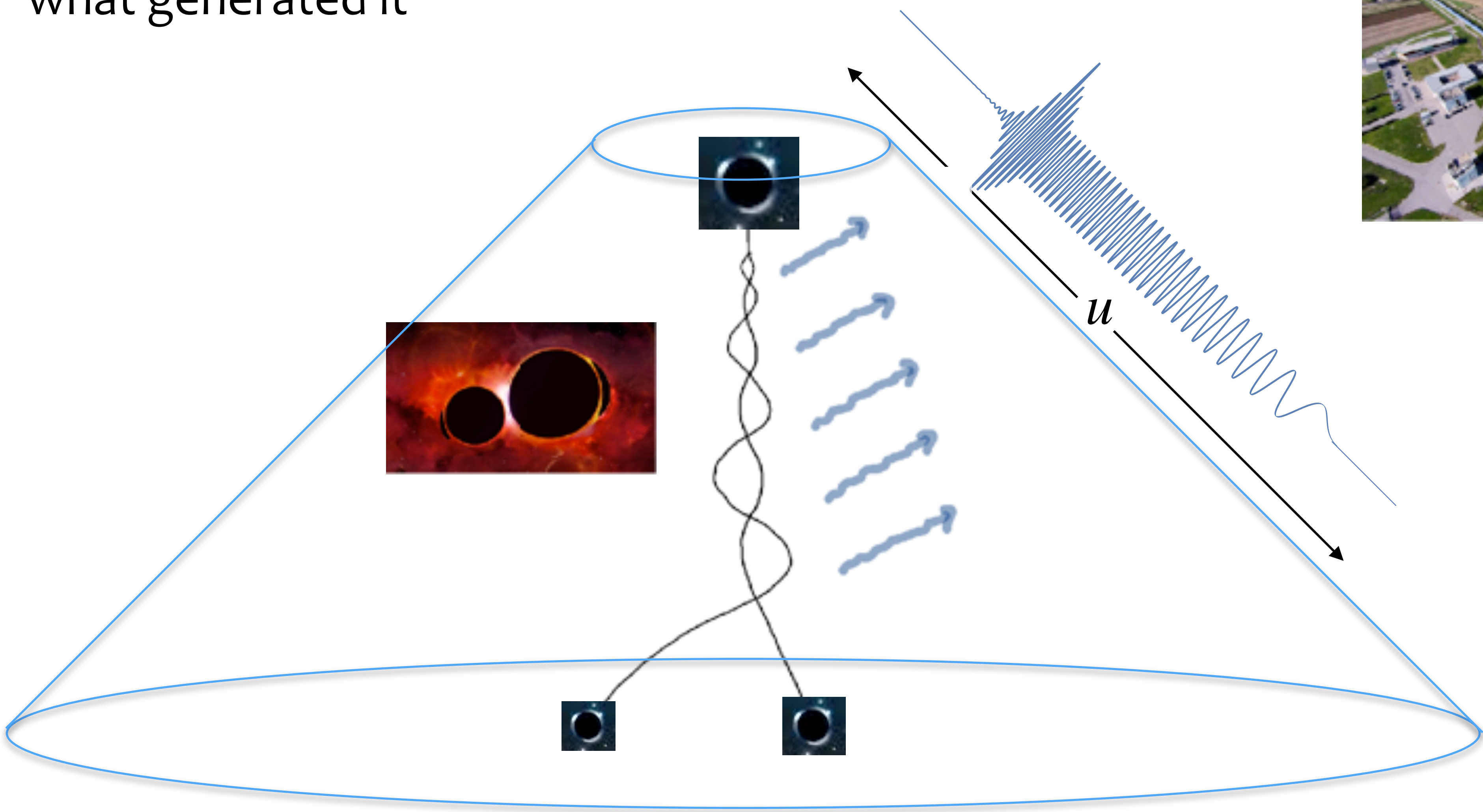


For a generic metric, no isometries, no conserved quantities.

This is always the case for radiating spacetimes. If GW are present, there are no global charges, but only surface charges, which change with the gravitational flux

Surface charges in GW astronomy

We pick up a GW signal, we would like to understand what generated it



How is the flux of GW related to properties of the source?

The study of asymptotic symmetries allows to understand the dynamical flux-balance laws in terms of Noether's theorem, and to approach physical questions with more tools

Surface charges: gravity

(*side remark*: description easier and more elegant using tetrads instead of the metric)

Lagrangian: $L = \sqrt{-g} R \epsilon$

Gauge transformations: $\delta_\xi L = d(i_\xi L)$ $\quad \delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu}$

Noether current: $j_\xi = \theta(\delta_\xi) - i_\xi L = dk_\xi + i_\xi C \approx dk_\xi$ $\quad k_{\xi\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma$

Noether charge: $Q_\xi = \int_\Sigma j_\xi \approx \int_{\partial\Sigma} k_\xi$

Without boundaries, or with trivial gauge transformations at the boundaries, no charges for the gravitational field
With boundaries and non-trivial gauge transformations we have access to infinitely many surface charges and their conservation laws

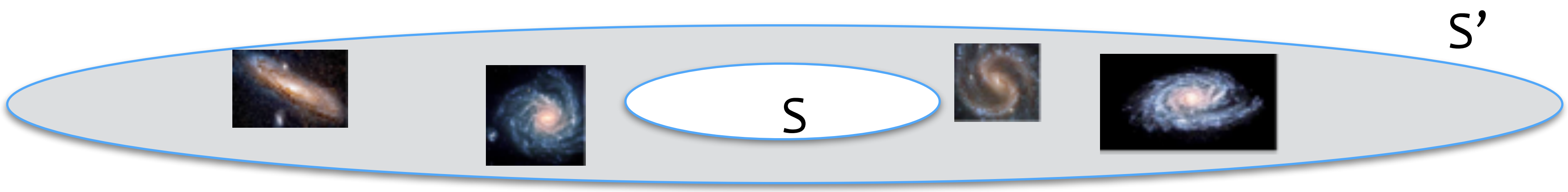
If the spacetime admits isometries, e.g. stationary ($\xi = \partial_\tau$) or axisymmetric ($\xi = \partial_\phi$), then

$$\int_\Sigma j_\xi \approx \int_\Sigma (T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T) \xi_\mu d\Sigma_\nu$$

Two key properties then hold:

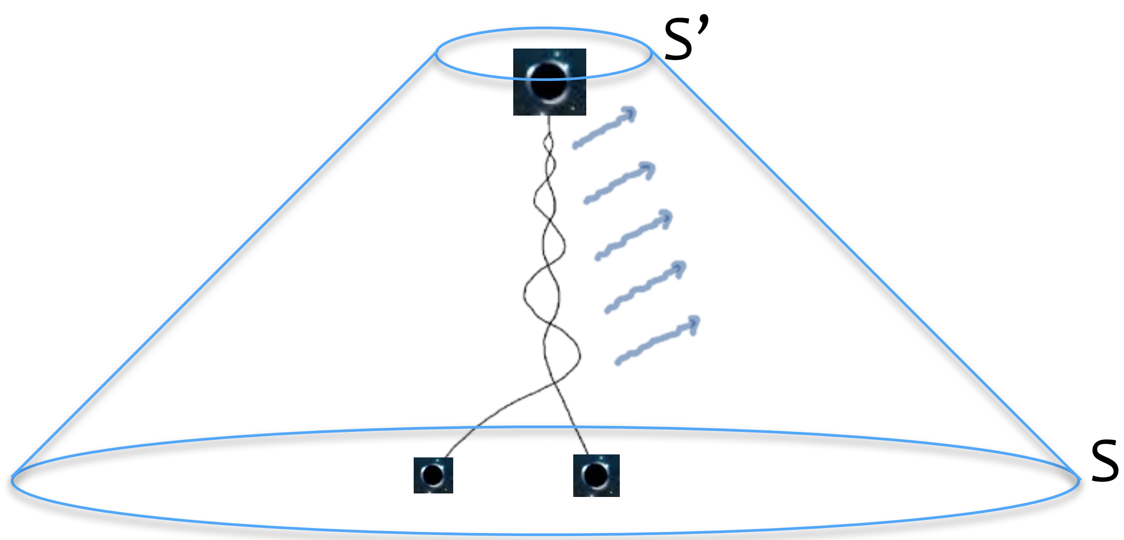
- The Noether charge does not depend on the surface in vacuum
- The variation of the Noether charge measures the matter energy-momentum content

Isometries versus radiation



$$Q_\xi[S'] - Q_\xi[S] \approx \int_\Sigma (T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)\xi_\mu d\Sigma_\nu$$

This is good, but does not hold in the presence of gravitational radiation.



$$Q_\xi[S'] - Q_\xi[S] = f(g, T, S, S')$$

If there is radiation, there is no isometry, no Killing vector: the Noether charge then depends on the surface chosen, hence the difference now depends on arbitrary choices such as coordinates and the 2-surface of integration, which have nothing to do with the gravitational dynamics.

Either the surfaces are somehow physically characterized, or these charges are not good observables

One common approach is to take the boundary all the way to infinity, so that spacetime is effectively empty, thus flat

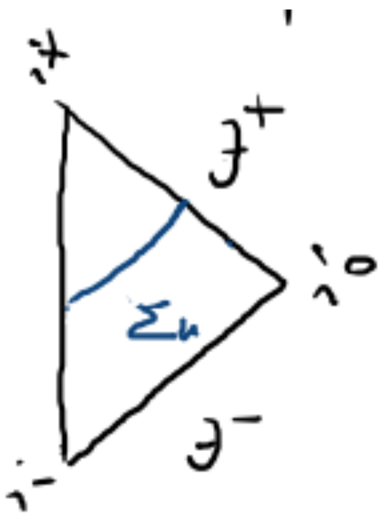
Two ways to go to infinity

Space-like infinity:

there is a natural choice of boundary conditions: $q_{ab} \stackrel{i^0}{\mapsto} \eta_{ab}$

<u>Residual diffeos</u>: the isometries of the flat metric! The Poincaré group, absent as global symmetry group of GR, emerges as asymptotic symmetry group of asymptotically flat metrics

Surfaces charges at spatial infinity: mass, angular momentum and more in general all Poincaré charges (ADM, Regge-Teitelboim, Beig-Murchadha, Ashtekar-Hansen, …)



Null infinity:

the same choice can be done, $q_{ab} \stackrel{\mathcal{I}}{\mapsto} \eta_{ab}$

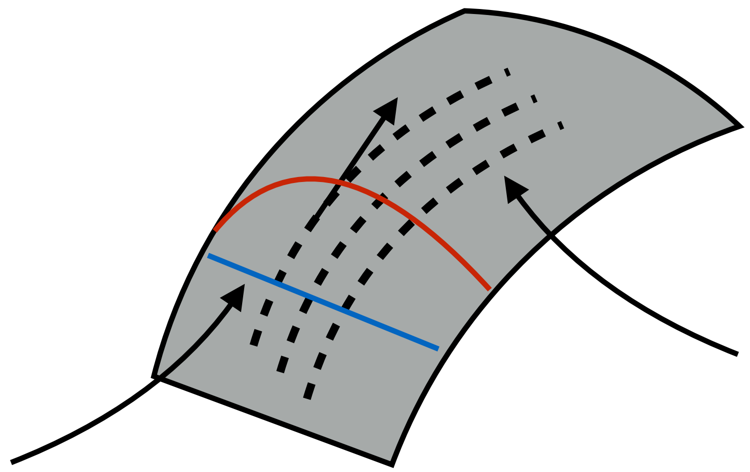
but does *not* select the Poincaré group

<u>Residual diffeos</u>: the isometries of a null slicing of the flat metric. This gives an infinite-dimensional group, known as the BMS group, which contains infinitely many copies of the Poincaré group, one for each cut of null infinity

Surfaces charges at null infinity: mass, angular momentum and more in general all BMS charges (Bondi-Metzer-Sachs, Newman-Penrose, Ashtekar-Streubel, Wald, …)

Why an infinite-dimensional symmetry group?

The big difference is that a null hyperplane has a degenerate metric:
there is no distinguished notion of Cartesian coordinate in the degenerate direction



transverse directions:
distances measured
by the round 2-sphere metric

zero distances along the null directions

the blue cut and the red cut are equivalently good observers

The extension is the freedom of making the translations of the Poincaré group not rigid, but depending on the point of the sphere: **supertranslations** $T(\theta,\phi)$

The ten-parameter Poincaré group is extended to the 6+a-function-on-S infinite-d BMS group

$$P^4 = \mathrm{SL}(2,\mathbb{C}) \ltimes T^4 \qquad \rightarrow \qquad BMS = \mathrm{SL}(2,\mathbb{C}) \ltimes R^S$$

(Pick a frame on the sphere, then the first 4 harmonics of T provide the translations wrt that frame)

Some details of the BMS group

Asymptotic expansion in Bondi coordinates:

$$ds^2 = -du^2 - 2dudr + r^2 \mathring{q}_{AB} dx^A dx^B$$ ← Leading order: Minkowski

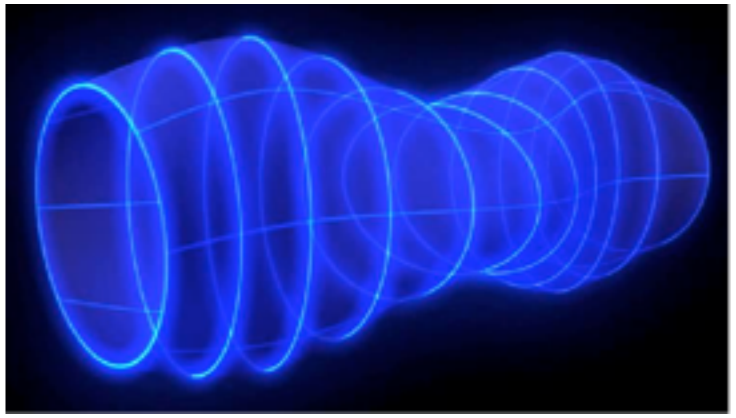
$$+\frac{2m}{r}du^2 + \bar{U}_A du dx^A + rC_{AB}dx^A dx^B$$

$$+\frac{\bar{P}_A}{r}dudx^A + \dots$$

mass aspect

angular momentum aspect

shear induced by a gravitational wave



The corresponding fall-off conditions:

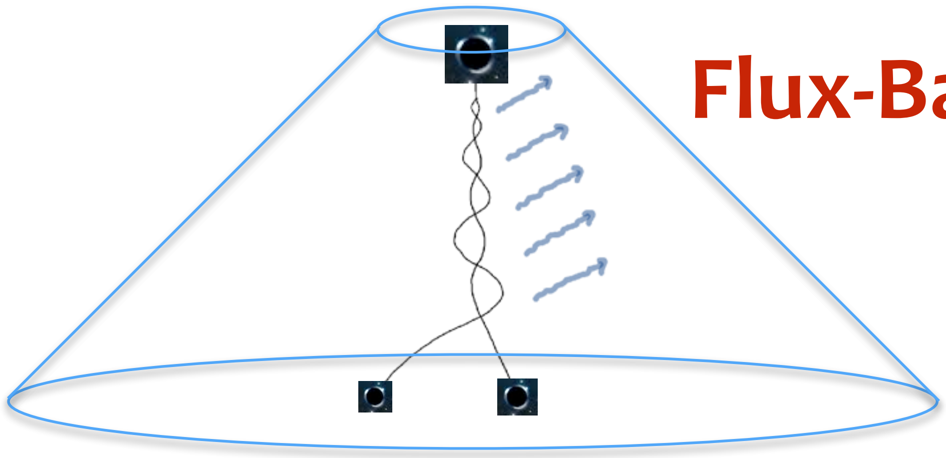
$$g_{ur} = -1 + \mathcal{O}(r^{-2}), \quad g_{uA} = \mathcal{O}(1), \quad g_{uu} = -1 + \mathcal{O}(r^{-1}), \quad q_{AB} = \mathring{q}_{AB} + \mathcal{O}(r^{-1})$$

are preserved by the following vector fields:

$$\bar{\xi}_{T,Y} := T\partial_u + Y^A\partial_A + \frac{1}{2}D\cdot Y(u\partial_u - r\partial_r)$$ with Y a CKV of the sphere

$$\to \quad BMS = \mathrm{SL}(2,\mathbb{C}) \ltimes R^S$$

In other words, the BMS group is generated by asymptotic Killing vectors, and the asymptotic metric has an infinite number of asymptotic symmetries



Flux-Balance laws at null infinity

Bondi-Metzer-Sachs, Newman-Penrose, Thorne, Ashtekar…

By going to null infinity, and picking up an enlarged infinite-d symmetry along the way, we are able to define surface charges that are independent of the surface and coordinate used, and that can be related to the gravitational flux

Caveat: for some diffeomorphisms (those not tangent to *S*), there is a shift between the Noether charge and the canonical generator of the symmetry

We can then:

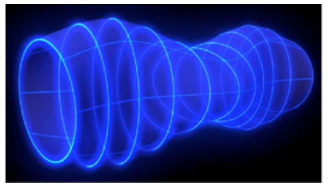
- identify those surface charges that in the stationary case reproduce the mass and angular momentum of black holes

$$M = \int_S m \qquad\qquad J = \int_S (\partial_\phi)^A P_A$$

- derive flux-balance laws directly from the Einstein's equations

$$\dot{m} = \frac{1}{4} D_A D_B \dot{C}^{AB} - \frac{1}{8} \dot{C}_{AB} \dot{C}^{AB} \qquad\qquad \dot{P}_A = \frac{1}{8} D_A (C_{BC} \dot{C}^{BC}) + \dots$$

Bondi energy-loss formula



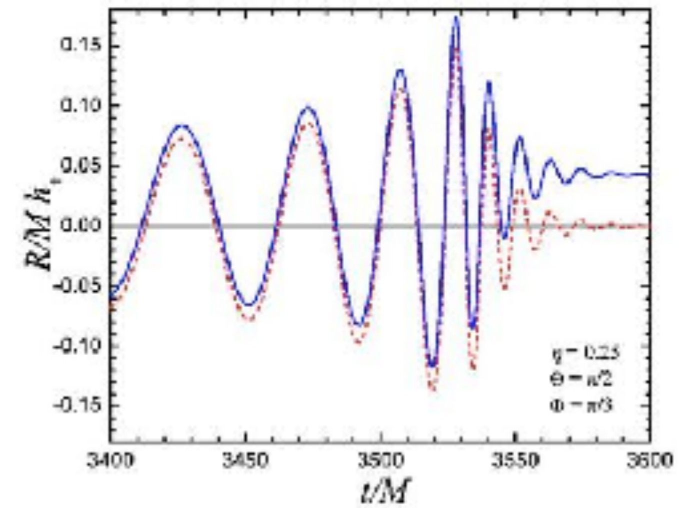
Angular momentum flux-balance law

These flux-balance laws are what allows us to reconstruct the physics of the source (i.e. the merging of BHs) from the observed signals (the GWs)

Strominger's infrared triangle

As in the e-m case, the boundary infinite-d symmetry associated with $T(\theta, \phi)$ is related to a memory effect: a permanent displacement of the shape of the detector

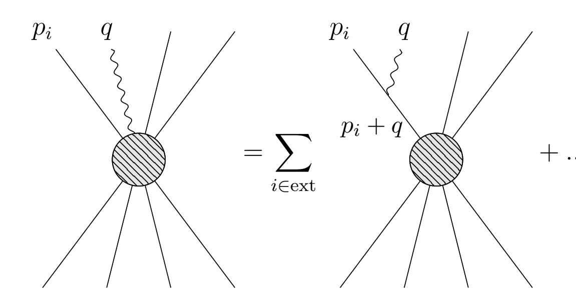
(Christodoulou '90s, Teukolsky, Nichols, Favata, …)



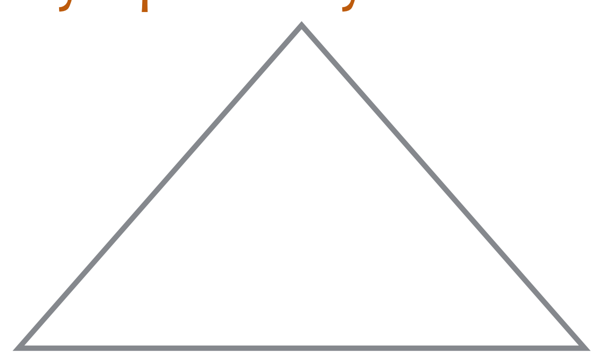
GW with memory

GW w/o memory

Again (Strominger et al) the Ward identities of the BMS symmetry reproduce Weinberg's soft theorem



asymptotic symmetries



memory effects

soft theorems

III - The Weyl BMS group

Motivations

Is the BMS symmetry the end of the story? Many reasons to think not!

- subleading soft theorems (Strominger, …) ⇨ gen. BMS group (Campiglia-Laddha, Compere-Fiorucci-Ruzziconi ’18)

- holography (Barnich, …) ⇨ extended BMS group (Barnich-Troessaert ’11)

- biggest symmetry algebra better quantization (Barnich, Grumiller, Freidel, …)

- the algebra of quasi-local observables is actually bigger (Flanagan et al, Ciambelli et al, Freidel…)

- … various further ideas (Hawking-Strominger-Perry, …)

A hierarchy of asymptotic symmetries

	Asymptotic background structure	*Symmetry parameters*	*Algebra*
Original BMS	round 2-sphere	$T(\theta,\phi)$, $Y^A(\theta,\phi)$ CKV	$\mathrm{SL}(2,\mathbb{C}) \ltimes R^S$
Extended BMS	round 2-sphere	$T(\theta,\phi)$, $Y^A(\theta,\phi)$ mero CKV	$\mathrm{Virasoro} \ltimes R^S$
Generalized BMS	scale factor	$T(\theta,\phi)$ $Y^A(\theta,\phi)$	$\mathrm{Diff}(S) \ltimes R^S$
Weyl BMS	foliation of *scri*	$T(\theta,\phi)$ $Y^A(\theta,\phi)$ $W(\theta,\phi)$	$(\mathrm{Diff}(S) \ltimes R^S) \ltimes R^S$

These extensions are quite challenging to achieve, and one has to deal with various technical points: holographic renormalization, charge-integrability, covariant phase space with anomalies, etc.

Some technical details of the BMSW group

We relax the original BMS conditions

$$g_{ur} = -1 + \mathcal{O}(r^{-2}), \quad g_{uA} = \mathcal{O}(1), \quad g_{uu} = -1 + \mathcal{O}(r^{-1}), \quad q_{AB} = \mathring{q}_{AB} + \mathcal{O}(r^{-1})$$

$$\bar{\xi}_{T,Y} := T\partial_u + Y^A\partial_A + \frac{1}{2}D\cdot Y(u\partial_u - r\partial_r) \qquad BMS = \mathrm{SL}(2,\mathbb{C}) \ltimes R^S$$

to:

$$g_{ur} = -1 + \mathcal{O}(r^{-2}), \qquad g_{uA} = \mathcal{O}(1), \qquad g_{uu} = \mathcal{O}(1), \qquad q_{AB} = \mathcal{O}(1)$$

$$\bar{\xi}_{(T,W,Y)} := T\partial_u + Y^A\partial_A + W(u\partial_u - r\partial_r) \qquad (\mathrm{Diff}(S) \ltimes R^S) \ltimes R^S$$

Recovering BMS as a subgroup:

0

$$\pounds_\xi g_{AB} = r^2(\pounds_Y \bar{q}_{AB} - 2W\bar{q}_{AB}) + O(r)$$

⇨ Y CKV

0

$$\pounds_\xi g_{uu} = (\pounds_Y + 2W)\bar{R} + 2\bar{\Delta}W + O(r^{-1})$$

⇨ $W = \frac{1}{2}D\cdot Y$

In particular this shows that the BMSW vectors are **not** asymptotic Killing vectors of the flat Minkowski metric

They provide asymptotic symmetries in a more general sense, wrt a *smaller* background structure

Advantages of the BMSW extension

- Includes all other symmetries as subcases
- matches the algebra of quasi-local observables
- disentangles diffeos and Weyl rescalings of the 2-sphere

A new, precise motivation to enlarge the asymptotic symmetry, uncovered from our work: there is a deep interplay between dynamics and the symmetry algebra

$$\{Q_\xi, Q_\chi\} = Q_{[\xi,\chi]} + \int_S \xi^\mu \chi^\nu G_{\mu\nu}$$

- The surface charges provide a representation of the algebra on-shell of the field equations
- Conversely, requiring a representation of the algebra imposes the Einstein's equations

  But how many of them? ***the larger the algebra, the more of the ten equations can be derived***

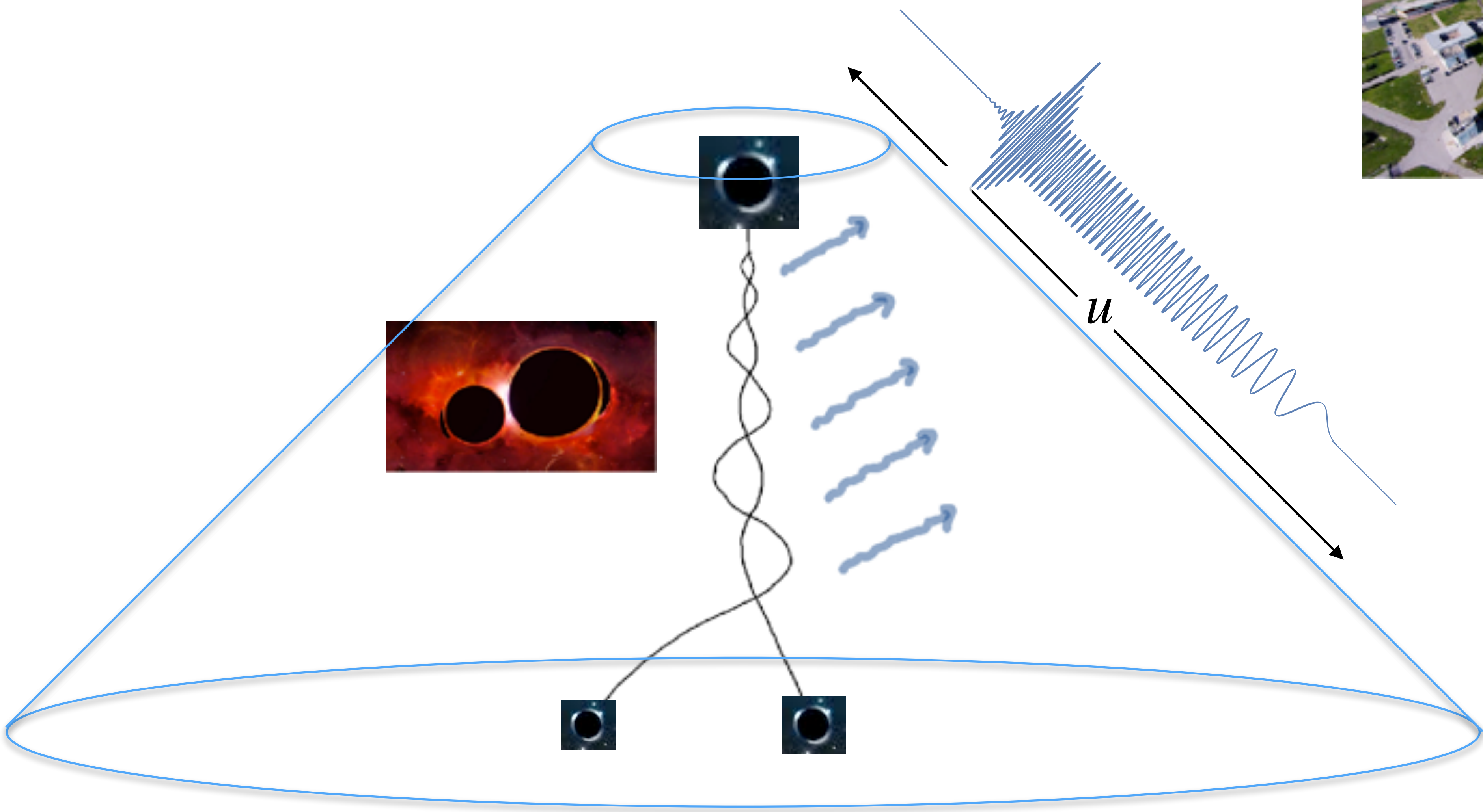
Original BMS : only 1 equation! (The Bondi energy loss)

Generalized BMS : 3 equations (The energy and angular momentum loss formulas)

Our new Weyl BMS : 8 equations

Representation of the charge algebra in phase space

Constructing this bracket representation of the charge algebra require solving a non trivial problem:



How do we construct a phase space when dofs are being radiated away?

Potential for physical predictions

Extending the symmetry also means potentially new memory effects

The extension from BMS to generalized BMS has already been shown to lead to new memory effects; *How about the Weyl BMS group?*

We don't know yet, but we have a few years to figure it out: memory is broadcasted to be observed in the next decade through piling up LIGO/Virgo data and through LISA

Why the question mark: New symmetries of GR **?**

We have identified a new class of symmetries associated with weaker mathematical b.c.,

but whether they are physically realised is still to be seen

In any case, it is an area of study that seats at a beautiful crossroad:

implications for quantum gravity

mathematical general relativity

study of asymptotic symmetries and surface charges

entanglement of subregions

potential physical predictions

links to a specific notion of holography

dual charges