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GLOBAL FLUCTUATIONS IN PHYSICAL SYSTEMS: A SUBTLE INTERPLAY BETWEEN SUM AND EXTREME VALUE STATISTICS

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Fluctuations of global additive quantities, like total energy or magnetization for instance, can in principle be described by statistics of sums of (possibly correlated) random variables. Yet, it turns out that extreme values (the largest value among a set of random variables) may also play a role in the statistics of global quantities, in a direct or indirect way. This review discusses different connections that may appear between problems of sums and of extreme values of random variables, and emphasizes physical situations in which such connections are relevant. Along this line of thought, standard convergence theorems for sums and extreme values of independent and identically distributed random variables are recalled, and some rigorous results as well as more heuristic reasonings are presented for correlated or non-identically distributed random variables. More specifically, the role of extreme values within sums of broadly distributed variables is addressed, and a general mapping between extreme values and sums is presented, allowing us to identify a class of correlated random variables whose sum follows (generalized) extreme value distributions. Possible applications of this specific class of random variables are illustrated on the example of two simple physical models. A few extensions to other related classes of random variables sharing similar qualitative properties are also briefly discussed, in connection with the so-called BHP distribution.

Keywords: Non-Gaussian fluctuations; Extreme value statistics; Correlated systems; Central limit theorem.

1. Introduction

The present review deals with the issue of fluctuations of global quantities and the possible relevance of extreme values in the statistics of these fluctuations. A physical quantity is called a global quantity, or global observable, when it is defined as the spatial sum of local quantities in a large enough system (or subsystem). For instance, the total energy and the total magnetization of a macroscopic system are global

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observables. Such quantities are very frequent in physics since on the one hand they are useful characterizations of macroscopic systems, and on the other hand most measurement apparatus have a resolution that is much larger than the scale of individual degrees of freedom. Hence from a practical viewpoint, most measured quantities turn out to be sums of local quantities, and thus fall into the category of global observables.

When measuring a global observable, one usually records a time signal, and the first quantity that can be determined from this signal is the mean value, which is a natural characterization of the system (assuming the system to be in a stationary state; extension to slow time evolution may also be considered). Then, to go beyond mean values, it is interesting to quantify fluctuations around the mean, for instance by determining the empirical variance of the signal. A more refined information is given by the full probability distribution of the fluctuations. Such a distribution is of particular interest since it may give information on the physics of the system. The main reason for this is the existence of the Central Limit Theorem (CLT), which states that a sum of a large number of independent and identically distributed (i.i.d.) random variables is distributed according to a Gaussian (or normal) distribution, provided that the second moment of each individual variable is finite – a more precise statement of the theorem will be given below.

The CLT, as any other theorem, relies on some assumptions that are of course not always true in physical systems. In particular, the second moment of the individual variables may not be finite, in which case other different asymptotic distributions (the so-called Lévy-stable laws) are reached in the limit of an infinite number of terms. This may happen in different physical contexts, like glasses and disordered systems,^{1,2,3,4,5,6,7} laser cooling experiments,^{8,9} turbulent flows,^{10,11,12} or blinking of nanocrystals,¹³ to quote only a few of them. Qualitatively, the departure from the Gaussian distribution is in this case due to the fact that a few terms become extremely large and dominate the sum. In contrast, when the second moment is finite, all terms within the sum are of the same order, and none of them play a dominant role.

Another situation of physical relevance, for which Gaussian distributions may not be found (even in the limit of an infinite number of terms), is when the variables that are summed are correlated. Intuitively, one may expect that the Gaussian distribution still holds when the correlations are “weak”, so that “strong” enough correlations may be necessary to prevent the distribution of the sum from converging to the Gaussian law. On the mathematical side, one must admit that a rigorous generic approach to this problem is a really difficult task and, to the best of our knowledge, no general results or criteria seem to be available. Hence correlated variables can only be treated case by case, leading to a (probably still incomplete) gallery of various limit distributions.

From a physicist viewpoint, a qualitative understanding of systems involving sums of correlated variables may be gained in some cases from simple scaling arguments, allowing one to guess whether the CLT (or an extension to it) is expected

to hold or not. The simplest of these arguments is an estimate of the number of “effectively independent” random variables. When this number diverges with the total number of terms in the sum, it is reasonable (at least to the physicist’s mind) to consider that the limit distribution should be Gaussian (if the second moment is finite). Still, a significant number of physical systems do not fulfill this condition, and fluctuations of global observables then display a non-Gaussian statistics, even in the infinite size limit. Indeed, a vast number of different non-Gaussian distributions have been reported in different contexts, like critical phenomena,^{14,15,16,17,18} width distribution of growing interfaces,^{19,20,21,22,23,24} turbulent flows,^{25,26,27,28,29} driven nonequilibrium systems,^{30,31,32,33} or different types of quantum systems,^{34,35,36} to give only a few examples. It is interesting to note that among these non-Gaussian distributions, asymmetric distributions rather similar to those appearing in the context of extreme value statistics (the statistics of the maximum or minimum value in a set of random variables) have been repeatedly found.^{37,38,39,40,41,42,43,44,45,46,47,48,49,50,51} It is thus natural to wonder whether this is a mere coincidence, or if there may be some generic reasons for such a similarity.

Altogether it turns out that, in a rather unexpected way, extreme values seems to play a role in the statistics of random sums, both for sums of broadly distributed random variables and for sums of correlated or non-identically distributed variables. The analysis of these two problems actually reveals that the role of extreme values is very different in both cases. Specifically, in the case of correlated variables, the similarity with extreme value statistics does not come from a dominant contribution of the largest terms, but rather from a natural, though not obvious, mapping between extreme values and sums of random variables.

Having this intricate picture in mind, we believed it would be relevant to treat essentially on the same footing the statistics of sums and of extreme values of random variables, before dealing with the different relationships arising between these fields. Accordingly, the paper is organized as follows. Some elementary statistical concepts are briefly recalled in Sect. 2. Standard mathematical theorems concerning the convergence to asymptotic distributions of both random sums and extreme values are presented in Sect. 3, in the case of independent and identically distributed variables. The case of dependent and of non-identically distributed variables is addressed in Sect. 4, paying particular attention to physical arguments and examples.

Let us emphasize that giving an exhaustive review of such a large topic as sums and extreme values of random variables, is obviously far beyond the scope of the present review paper. Accordingly, the latter has the more modest aim of giving the reader an overall (though incomplete) picture, from a physicist perspective, of the basic results and issues in these fields, with specific emphasis on some recent results providing unexpected connections between them. In this spirit, we tried to make mathematical statements as precise as possible, while also leaving room for numerous physical discussions and examples.

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2. Some probabilistic concepts useful in physics

2.1. *Random variables: dependence, joint probability and spatial organization*

One of the main goals of statistical physics is to study the macroscopic properties of models defined by some microscopic interaction rules between "particles", or more generally, microscopic degrees of freedom. The paradigm behind this approach is the belief that there exists, at least for some classes of systems, general mechanisms for going from the microscopic rules to a macroscopic behavior, so that the knowledge gained from the study of specific models goes beyond a collection of particular results. In this section, we would like to go from the physicist's viewpoint to the mathematician's one: this will be useful in the rest of the present article, and might also illustrate the way mathematics and physics interact in statistical mechanics.

Let us start by one of the archetypal model of statistical physics, namely the Ising model. On each site $i = 1, \dots, N$ of a D -dimensional square lattice,^a a spin variable can take two possible values $s_i = \pm 1$. Each spin s_i interacts with its nearest neighbor and with an external magnetic field h , so that the total energy of a configuration $s \equiv \{s_i\}_{i=1\dots N}$ is given by the celebrated Hamiltonian:

$$\mathcal{H}[s] = -K \sum_{\langle i,j \rangle} s_i s_j - h \sum_{i=1}^N s_i, \quad s \equiv \{s_i\}_{i=1\dots N}. \quad (1)$$

The summation in the first term is over all pairs of nearest neighbor sites i and j . A global observable a statistical physicist may be interested in is the magnetization $m[s]$ of a given configuration:

$$m[s] = \frac{1}{N} \sum_{i=1}^N s_i. \quad (2)$$

In the canonical ensemble, the average of this quantity at a given inverse temperature $\beta = 1/k_B T$, is defined by

$$\langle m \rangle = \sum_s P[s] m[s], \quad P[s] = \frac{1}{\mathcal{Z}} e^{-\beta \mathcal{H}[s]}. \quad (3)$$

The normalization constant \mathcal{Z} is the canonical partition function of the model. The function $P[s]$ is then the probability of a given configuration s . Most often, it is not an easy task to explicitly compute the mean magnetization. The problem comes from the nearest neighbor interaction, which induces statistical correlations between sites i and j . Spins are independent only in the specific case $K = 0$, when no interactions are present. In this case one easily obtains $\langle m \rangle = \tanh(\beta h)$.

The above physical example could be rephrased in a more mathematical language. In probabilistic terms, a spin configuration s forms a set of random variables

^aThroughout the article, we generically denote as N the number of random variables considered.

(s_1, \dots, s_N) , and the probability $P[s]$ of a configuration is denoted as the joint probability $J_N(s_1, \dots, s_N)$ of the random set. The random set is fully characterized by its joint probability, which is the probability to obtain the particular set (s_1, \dots, s_N) from a realization of the random variables. From this joint probability one can define the marginal probability density $p_i(s_i)$ associated to the random variable s_i , by summing $J_N(s_1, \dots, s_N)$ over all the remaining variables:

$$p_i(s_i) = \sum_{\{s_j = \pm 1, j \neq i\}} J_N(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_N). \quad (4)$$

The N random variables s_i are independent if and only if the joint probability can be factorized as the product of the marginal distributions $p_i(s_i)$,

$$J_N(s_1, \dots, s_N) = \prod_{i=1}^N p_i(s_i). \quad (5)$$

In the Ising model, and more generally in canonical equilibrium systems, there is an obvious link between the joint probability and the Hamiltonian:

$$J_N(s_1, \dots, s_N) = \frac{1}{Z} \exp[-\beta \mathcal{H}(s_1, \dots, s_N)], \quad (6)$$

and the non-interacting case ($K = 0$) corresponds to independent variables in the probabilistic view:

$$J_N(s_1, \dots, s_N) = \frac{1}{Z} \prod_{i=1}^N e^{-\beta h \sigma_i}. \quad (7)$$

This elementary example shows how problems of statistical physics can be recast into a probabilistic language. Let us point out that in physics, in contrast to what happens in mathematics, a meaning is given to the random quantities, and to their labels.⁵² It introduces some key concepts from the physicist's viewpoint such as a distance (spatial or temporal) and a space dimensionality. This is the reason why the description of the dependence of random variables in terms of the joint probability is somehow too rich for the physicist, who often prefers a simpler characterization through the two-point correlation function:

$$C_{ij} = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle, \quad (8)$$

which emphasizes the spatial structure of the model. Under the assumption of homogeneity and isotropy, an even more 'compact' information can be obtained from a correlation length or time, defined as the characteristic scale of the correlation function (8),

$$C_{ij} = C_0 f\left(\frac{r_{ij}}{\xi}\right), \quad (9)$$

where f is a dimensionless function and r_{ij} is the euclidean distance between sites i and j . Quite often, a lot of information on a physical system can be gained from the behavior of the correlation length with temperature, magnetic field or other control parameters.

2.2. Concept of asymptotic distributions

As already pointed out, the specificity of physics (and of other sciences aiming at describing real systems in a mathematical language) is to give an interpretation, in connection with the real world, to the mathematical objects involved in the proposed description. From the physicist's point of view, probability theory provides various theorems that are kinds of "reasoning shortcuts", ready to be used in a given physical context. Among the most useful theorems are the convergence theorems, which will be presented in Sect. 3. In the present section, we recall elementary definitions and probabilistic tools. Our aim is not to present in a rigorous way probability concepts but to introduce practical tools that are needed in the following sections.

We already considered discrete random variables, like spin variables s , in the previous section. Discrete random variables are most often characterized by the probability $P[s]$ of each configuration s . In contrast, when considering continuous random variables, several probabilistic tools may be used depending on the context. From the physicist's point of view, a continuous random variable^b X is characterized by a non-negative function $p(x)$, the probability density function (PDF), or probability distribution, of X . This PDF is such that the cumulative probability distribution $F(x)$ can be expressed as

$$\text{Prob}(X \leq x) \equiv F(x) = \int_{-\infty}^x dx' p(x'). \quad (10)$$

The random variable X can equivalently be described by its characteristic function $\chi(\omega)$, defined as the Fourier transform of $p(x)$:

$$\chi(\omega) = \int_{-\infty}^{+\infty} dx p(x) e^{-i\omega x} = \langle e^{-i\omega x} \rangle_p.$$

Two random variables have the same characteristic function if and only if they have the same PDF. Since the PDF of the sum of two independent random variables x_1 and x_2 is the convolution product $p_1 \star p_2$ of their PDF, the characteristic function of the sum is the product $\chi_1 \chi_2$ of the characteristic functions of x_1 and x_2 . This property makes the characteristic function a very useful tool.⁵³

Once these mathematical objects are defined, let us ask the following question. Consider a sequence of N random variables $\{x_k\}_{k=1\dots N}$, distributed according to a joint probability distribution $J_N(x_1, \dots, x_N)$. Let us then define another random variable, say $y_N = \phi(x_1, \dots, x_N)$, where ϕ is a given arbitrary function. The question we ask is whether it is possible to find a sequence of numbers $\{(a_N, b_N > 0)\}$ such that the distribution function of the random variable $z_N = (y_N - a_N)/b_N$ converges toward a PDF $p_\infty(z)$, where $p_\infty(z)$ is a proper PDF (not concentrated at one point) when N goes to infinity. In such case, $p_\infty(z)$ is called the asymptotic, or

^bIn the rest of the paper, we shall most of the time use the same notation for a random variable X and its value x .

limit, distribution of z , and by extension of y . The joint probability distribution $J_N(x_1, \dots, x_N)$ is said to belong to the domain of attraction $\mathcal{D}(p_\infty)$ of p_∞ . In the case of i.i.d. random variables, for which the joint distribution factorizes as $J_N(x_1, \dots, x_N) = \prod_{i=1}^N p(x_i)$, one says that the marginal distribution $p(x)$ belongs to the domain of attraction $\mathcal{D}(p_\infty)$.

For a generic function $\phi(x_1, \dots, x_N)$, it is extremely difficult to identify the asymptotic distribution (when it exists) and the corresponding domain of attraction. However, two particular examples have been extensively studied by mathematicians: the case of sums of random variables,^{54,55,56,57,58} $\phi(x_1, \dots, x_N) = \sum_{i=1}^N x_i$, and the case of extreme values, $\phi(x_1, \dots, x_N) = \max(x_1, \dots, x_N)$.^{59,60,61,62,63,64} As already pointed out in the introduction, we shall focus in this article on these two specific cases, and see that under additional assumptions, mathematics provides some very interesting results.

As a side remark, let us note that words like “domain of attraction” and “limit function” might ring a bell to the physicist: it is indeed reminiscent of the renormalization group language. This is actually not a mere coincidence, as pointed out for instance by Jona-Lasinio.⁶⁷ One can actually see the renormalization group procedure as a way to compute limit distributions in physical situations where standard mathematical results may not be applicable. We shall come back to this point in Sect. 4.1.5.

3. Asymptotic distributions of sums and extremes of i.i.d. random variables

3.1. Statistics of random sums and Gaussian distribution

3.1.1. Standard Central Limit Theorem for i.i.d. random variables

Let us consider a set of random variables $\{x_i\}_{i=1, \dots, N}$, and let us denote their sum, which is also a random variable, as y_N :

$$y_N = \phi(x_1, \dots, x_N) = \sum_{i=1}^N x_i. \quad (11)$$

This new random variable could *a priori* be described by an arbitrary PDF. However, it turns out that the Gaussian (or normal) distribution appears in such a large collection of phenomena that it has been somehow considered as “universal”. This belief led Henry Poincaré to his famous comment:⁶⁸ “*All the world believes it firmly, since mathematicians imagine that it is a fact of observation and the observers that it is a mathematical theorem.*” In this section we summarize the main standard theorems, and comment on their applications in physics.

The case of sums of i.i.d. random variables is clearly the simplest one. In this case, there is only one single free distribution in the problem:

$$J_N(x_1, \dots, x_N) = \prod_{i=1}^N p(x_i), \quad (12)$$

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and it is possible to establish the following standard version of the Central Limit Theorem.^{54,57}

Theorem 1. *Let $\{x_i\}_{i=1,\dots,N}$ be a sequence of N random variables, independent and identically distributed, with a finite mean value m , and a finite variance σ . Let us define $y_N = \sum_{i=1}^N x_i$, $a_N = Nm$, and $b_N = \sigma\sqrt{N}$. Then the random variable*

$$z_N = \frac{y_N - a_N}{b_N}. \quad (13)$$

converges in law when $N \rightarrow \infty$ to the normal distribution $\mathcal{N}_{0,1}$.

The normal distribution $\mathcal{N}_{0,1}$ is the Gaussian distribution with zero mean and unit variance

$$\mathcal{N}_{0,1}(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \quad (14)$$

Accordingly, one sees that three essential hypotheses are needed in order to apply the Central Limit Theorem (at least in its standard form):

- the variables x_i are independent.
- the variables x_i are identically distributed.
- the second moment of x_i is finite (the finiteness of the mean value and of the variance then follows).

When at least one of these conditions is not fulfilled, the standard form of the Central Limit Theorem is no longer valid. We see that given the fact that the variables are i.i.d., this theorem applies to a large number of probability distributions, as the only requirement is the existence (i.e., finiteness) of the second moment. In other words one could say that the basin of attraction of the normal distribution is ‘large’.

It is possible to extend slightly the previous theorem to include some distributions without second moment. The CLT for i.i.d. random variables in its general form is given below.⁵⁴

Theorem 2. *Let $\{x_i\}_{i=1..N}$ be a sequence of N random variables, independent and identically distributed according to the probability distribution $p(x)$. Then $p(x)$ belongs to the attraction basin of the normal law if and only if*

$$\lim_{X \rightarrow \infty} X^2 \frac{\int_{|x|>X} dx p(x)}{\int_{|x|<X} dx x^2 p(x)} = 0. \quad (15)$$

As an illustration of this extension, let us consider the case of a power-law probability density $p(x)$:

$$p(x) = \frac{\alpha x_0^\alpha}{x^{1+\alpha}} \Theta(x - x_0), \quad (x_0 > 0). \quad (16)$$

If $\alpha > 2$, the second moment of the distribution is finite, and Theorem 1 applies. In contrast, if $\alpha \leq 2$, the second moment is infinite, and Theorem 1 no longer holds. However, for $\alpha = 2$ we have

$$X^2 \frac{\int_{|x|>X} dx p(x)}{\int_{|x|<X} dx x^2 p(x)} = \frac{1}{2 \ln(X/x_0)} \rightarrow 0, \quad X \rightarrow \infty. \quad (17)$$

Theorem 2 then shows that this distribution is actually in the attraction basin of the Normal law. Moreover similar calculations for $\alpha < 2$ show that

$$X^2 \frac{\int_{|x|>X} dx p(x)}{\int_{|x|<X} dx x^2 p(x)} = \frac{2-\alpha}{\alpha} \frac{X^{2-\alpha}}{X^{2-\alpha} - x_0^{2-\alpha}} \rightarrow \frac{2-\alpha}{\alpha}, \quad X \rightarrow \infty. \quad (18)$$

These argument can be extended in a straightforward way to distributions $p(x)$ with a power-law tail, $p(x) \sim x^{-1-\alpha}$, $x \rightarrow \infty$. Therefore probability densities with power-law tails with $\alpha < 2$ do not belong to the attractive basin of the Gaussian distribution.

Before discussing extreme value statistics, as well as several extensions of the Central Limit Theorem, we would now like to briefly comment on the usefulness of the CLT in physics.

3.1.2. Examples of applications of the Central Limit Theorem in physics

The Central Limit Theorem is one of the cornerstones of statistical physics, giving general results about fluctuations of global quantities in systems where correlations among the different degrees of freedom can be neglected. As an illustration, let us consider a generic system with degrees of freedom $\{q_i\}$, $i = 1, \dots, N$, described by a Hamiltonian which can be written as the sum of one-body Hamiltonians:

$$\mathcal{H}(\{q_i\}) = \sum_{i=1}^N \mathcal{H}_1(q_i) \quad (19)$$

where $\mathcal{H}_1(q_i)$ only depends on the single variable q_i (q_i may be a real variable or a vector depending on the system considered). At statistical equilibrium in the canonical ensemble, the joint distribution of the variables $\{q_i\}$ is given by

$$J(\{q_i\}) = \frac{1}{\mathcal{Z}_1^N} \exp\left(-\beta \sum_{i=1}^N \mathcal{H}_1(q_i)\right) = \prod_{i=1}^N \frac{1}{\mathcal{Z}_1} e^{-\beta \mathcal{H}_1(q_i)}. \quad (20)$$

The partition function factorizes as a product of N one-body partition functions \mathcal{Z}_1 , given by:

$$\mathcal{Z}_1 = \int_{-\infty}^{\infty} dq e^{-\beta \mathcal{H}_1(q)} \quad (21)$$

Accordingly, the different degrees of freedom are i.i.d. random variables. In many cases, the variance of q_i is finite, so that the CLT can be applied to the sum $S = \sum_{i=1}^N q_i$. Once suitably rescaled, this sum thus has a Gaussian statistics in the large

N limit, independently of the specific form of the one-body Hamiltonian (as long as the variance of q_i remains finite). Consider for instance a paramagnet consisting of an assembly of N Ising spins $s_i = \pm 1$, and with a Hamiltonian

$$\mathcal{H}_{\text{para}}(\{s_i\}) = - \sum_{i=1}^N h s_i \quad (22)$$

where h is the external magnetic field. The variance of s_i is obviously finite since s_i is bounded. From the CLT, the total magnetization $M = \sum_{i=1}^N s_i$ is distributed according to a Gaussian law in the thermodynamic limit $N \rightarrow \infty$.

Coming back to the general Hamiltonian $\mathcal{H}(\{q_i\})$, another issue of physical interest is to determine the fluctuations of the total energy of the system, that is of the value of $\mathcal{H}(\{q_i\})$. One then needs to make a change of variables, introducing $\varepsilon_i = \mathcal{H}_1(q_i)$, in order to rewrite the total energy as the sum $E = \sum_{i=1}^N \varepsilon_i$. From Eq. (20), the variables ε_i are also i.i.d. random variables, with distribution

$$p(\varepsilon_i) = \frac{1}{\mathcal{Z}_1} e^{-\beta \varepsilon_i} |\mathcal{J}(\varepsilon_i)| \quad (23)$$

where $\mathcal{J}(\varepsilon_i)$ is the Jacobian of the transformation. If the variance of ε_i is finite, then the CLT can be applied to the total energy E which then follows a Gaussian statistics. As a simple example, let us consider a classical gas of independent particles. In the absence of external field, the Hamiltonian reduces to the kinetic energy:

$$\mathcal{H}_{\text{gas}}(\{\mathbf{p}_i\}) = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} \quad (24)$$

The distribution of kinetic energy per particle $\varepsilon_i = \mathbf{p}_i^2/2m$ is then, in D dimensions,

$$p(\varepsilon_i) = \frac{\beta^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \varepsilon_i^{\frac{D}{2}-1} e^{-\beta \varepsilon_i}, \quad (25)$$

where $\Gamma(t) = \int_0^\infty du u^{t-1} e^{-u}$ is the Euler Gamma function. The second moment is finite, so that from the CLT, the total energy follows a Gaussian statistics.

As we have seen, a large class of observables obey a Gaussian statistics in equilibrium systems, provided that the energy of the coupling terms in the Hamiltonian can be neglected. When this is not the case, it is often possible at a heuristic level to split the system into essentially independent “blocks” that play the same role as the independent degrees of freedom in the above description. Hence, the CLT is often effectively valid in physical systems, although the strict hypotheses underlying it may not be fulfilled. These issues are discussed in more details in Sect. 4.

Note also that many other examples of the application of the CLT in statistical physics could be given, where the variables to be summed are not described by a Hamiltonian, specifically when considering out-of-equilibrium systems. This is the case for instance for the (algebraic) distance travelled by a one-dimensional random walk. A possible physical realization is to follow the coordinate along a given axis of a Brownian particle in a fluid. If the steps of the walk (i.e., the distance between two

successive collisions) are independent random variables with finite second moment, then the coordinate of the particle at a given large time is distributed according to a Gaussian law.

3.2. Extreme value statistics of i.i.d. random variables

3.2.1. Universality classes and asymptotic distributions

Another example of convergence theorem is given by the mathematical theory of extreme value statistics, that is the maximum or minimum value in a set of random variables. Following Sect. 2.2 this situation corresponds to the function $\phi(x_1, \dots, x_N) = \max(x_1, \dots, x_N)$. Extreme value statistics has found applications in many different fields, like physics of disordered systems,⁸⁵ chemical fracture,⁸⁶ hydrology,⁸⁷ seismology,⁸⁸ or finance,^{89,90} to quote only some of them.

For the sake of simplicity we shall restrict ourselves to the case of maximal values, but equivalent results are available for minimal values, since a minimum can be converted into a maximum by simply changing the sign of the random variable. The problem is the following. Let $\{x_i\}_{i=1\dots N}$ be a set of N i.i.d. random variables, whose common cumulative distribution is $F(x)$ (i.e., the probability that the random variable is less than the given value x). From each realization of the set, one can define a new random variable

$$z_N = \max(x_1, \dots, x_N). \quad (26)$$

As by definition $x_i \leq z_N$ for all i , the probability $F_N(z)$ that the maximum is less than a value z (that is, the cumulative distribution of the maximum) is simply, from the i.i.d. property,

$$\begin{aligned} F_N^{\max}(z) &\equiv \text{Prob}(\max(x_1, \dots, x_N) < z) \\ &= \text{Prob}(\forall i, x_i < z), \end{aligned} \quad (27)$$

so that one has

$$F_N^{\max}(z) = F(z)^N. \quad (28)$$

A natural question is to try to determine the asymptotic cumulative distribution of this maximum value z_N (if it exists), with the hope that it does not depend on all the details of the original cumulative distribution F . More precisely, one should wonder whether it would be possible to find a sequence $\{a_N, b_N\}$ such that

$$\lim_{N \rightarrow \infty} F_N^{\max}(a_N + b_N x) = H(x), \quad (29)$$

where H is a non-degenerate cumulative probability function to be determined as well. We then have the following results.^{60,61}

Theorem 3. *The asymptotic cumulative distribution $H(x)$ of a set of i.i.d. random variables $\{x_i\}_{i=1,\dots,N}$ is necessarily (up to a shift and a dilatation of the variable x) of the form $H_{f,\mu}(x)$, $H_{w,\mu}(x)$ or $H_g(x)$ with*

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- $H_{f,\mu}(x) = \exp(-x^{-\mu})$ for $x > 0$ and 1 for $x \leq 0$ (Fréchet distribution);
- $H_{w,\mu}(x) = \exp(-(-x)^\mu)$ for $x < 0$ and 1 for $x \geq 0$ (Weibull distribution);
- $H_g(x) = \exp(-e^{-x})$ (Fisher-Tippett-Gumbel or Gumbel distribution).

In the previous expressions, $\mu > 0$ is a parameter, called the extreme value index, depending on the parent cumulative probability distribution $F(x)$. Note that a more compact parameterization has been proposed by von Mises⁹¹. Up to a dilatation and a shift of the variable x , the asymptotic extreme value distributions $H_{f,\mu}(x)$, $H_{w,\mu}(x)$ and $H_g(x)$ can be reformulated as $H_\gamma(x)$, with $a > 0$ and b real, where

$$H_\gamma(x) = \exp\left(-\left(1 + \gamma x\right)^{-1/\gamma}\right), \quad 1 + \gamma x > 0. \quad (30)$$

The parameter γ is real and the value $\gamma = 0$ in the right-hand side should be interpreted as $\exp(-e^{-x})$, then corresponding to the Gumbel distribution. The case $\gamma > 0$ corresponds to the Fréchet distribution, while $\gamma < 0$ stands for the Weibull case.

Hence the parameter μ in Theorem 3 is related to the parameter γ in Eq. (30) through $\mu = 1/\gamma$ if $\gamma > 0$ (Fréchet distribution) and $\mu = -1/\gamma$ if $\gamma < 0$ (Weibull distribution). Now we have to specify the attraction basin $\mathcal{D}(H_\gamma)$ for each of the previous distributions. This is the purpose of the following theorem.⁶⁴

Theorem 4. *Let $F(x)$ and $H(x)$ be non-degenerate cumulative distribution functions, such that for some constants $a_N > 0$ and b_N real*

$$\lim_{N \rightarrow \infty} F(a_N + b_N x)^N = H(x).$$

Then H is up to a location and scale shift an extreme value distribution H_γ , and F belongs to the attraction basin $\mathcal{D}(H_\gamma)$ of the distribution H_γ . We also define $\tilde{F}(z) = 1 - F(z)$, which satisfies $\lim_{z \rightarrow \infty} \tilde{F}(z) = 0$. Let $\omega_F = \sup\{z : \tilde{F}(z) > 0\}$. We then have:

- *F belongs to the attraction basin $\mathcal{D}(H_{f,\mu}) = \mathcal{D}(H_{\gamma=1/\mu})$ if, and only if, $\omega_F = +\infty$ and for all $x > 0$,*

$$\lim_{t \rightarrow \infty} \frac{\tilde{F}(tx)}{\tilde{F}(t)} = x^{-\mu}, \quad \mu > 0; \quad (31)$$

- *F belongs to the attraction basin $\mathcal{D}(H_{w,\mu}) = \mathcal{D}(H_{\gamma=-1/\mu})$ if, and only if, ω_F is finite and for all $x > 0$,*

$$\lim_{t \rightarrow \infty} \frac{\tilde{F}\left(\omega_F - \frac{1}{tx}\right)}{\tilde{F}\left(\omega_F - \frac{1}{t}\right)} = x^{-\mu}, \quad \mu > 0; \quad (32)$$

- *F belongs to the attraction basin $\mathcal{D}(H_g) = \mathcal{D}(H_{\gamma=0})$ if, and only if, for some finite ω_0 the integral $\int_{\omega_0}^{\omega_F} dz \tilde{F}(z)$ is finite and for all $x > 0$,*

$$\lim_{t \rightarrow \infty} \frac{\tilde{F}(t + xR(t))}{\tilde{F}(t)} = e^{-x}, \quad (33)$$

with $R(t)$ defined by

$$R(t) = \frac{1}{\tilde{F}(t)} \int_t^{\omega_F} dz \tilde{F}(z), \quad (34)$$

Note that it is also customary to speak about classes instead of basins of attraction, in the context of extreme value statistics. For instance, the exponential distribution is said to be in the Gumbel class. Let us now try to give an intuitive interpretation of Theorem 4. Essentially, it means that when $\tilde{F}(z)$, namely the probability that the random variable is greater than z , decays like a power law at large z , $\tilde{F}(z) \sim z^{-\mu}$ with $\mu > 0$, the limit distribution is the Fréchet one with parameter μ . When z is a bounded variable, that is $\tilde{F}(z) = 0$ for $z > \omega_F$, and if $\tilde{F}(z) \sim (\omega_F - z)^\mu$, $z \rightarrow \omega_F^-$ with $\mu > 0$, then the asymptotic distribution is the Weibull one with parameter μ . Finally, if z is unbounded, but if $\tilde{F}(z)$ decays faster than any power law when $z \rightarrow \infty$, the asymptotic distribution is the Gumbel one. Note however that the theorem also leads to the Gumbel distribution in some cases of bounded variables, like for instance $\tilde{F}(z) \sim \exp[-c(\omega_F - z)^{-\alpha}]$, $z \rightarrow \omega_F^-$, where α and c are positive constants.

As an illustration of how these limit laws can be derived, let us consider the simple example of the exponential distribution, for which the cumulative probability distribution is $F(z) = [1 - \exp(-\lambda z)]$ for $z > 0$ and $F(z) = 0$ for $z \leq 0$. In this case, the cumulative distribution $F_N^{\max}(z)$, i.e., the probability that the largest value in the set in N variables drawn from F is smaller than z , reads

$$F_N^{\max}(z) = F(z)^N = (1 - e^{-\lambda z})^N \quad (35)$$

On general grounds, one wishes to find a rescaling $z = a_N + b_N x$ such that the distribution of x converges to a well-defined limit when N goes to infinity. In the present case, it is rather easy to see that a correct choice for the rescaling is $a_N = (\ln N)/\lambda$ and $b_N = 1/\lambda$, yielding

$$F_N^{\max}(a_N + b_N x) = \left(1 - \frac{e^{-x}}{N}\right)^N \rightarrow \exp(e^{-x}) \quad (36)$$

when $N \rightarrow \infty$. Hence the distribution $F_N^{\max}(a_N + b_N x)$ readily converges to the Gumbel distribution $H_g(x)$. Clearly, the rigorous derivation of the asymptotic distributions in the case of an arbitrary $F(z)$ is much more difficult. Yet, the main ideas of the derivation are already sketched in this simple example.

3.2.2. Asymptotic distributions for the k^{th} largest value

Up to now we only considered the asymptotic distribution of the largest value within a set of N i.i.d. random variables, in the limit $N \rightarrow \infty$. A further natural question would then be to know how the k^{th} largest value in the random set is distributed. Note that here k is independent of N , the number of elements in the set; the case where $k = k(N)$, sometimes referred to as order statistics, could also be studied,⁶¹

but is outside the scope of the present review. Let us consider the cumulative probability $F_{k,N}(x)$ of the k^{th} largest value $X_{k,N}$ in a set of N i.i.d. random variables. Then relation (28) can be generalized to obtain⁶⁰

$$F_{k,N}(x) = \text{Prob}(X_{k,N} \leq x) = \sum_{j=0}^{k-1} \frac{N!}{j!(N-j)!} F(x)^{N-j} [1 - F(x)]^j, \quad (37)$$

Keeping fixed the index k while $N \rightarrow \infty$, it turns out that the asymptotic form of $F_{k,N}$ is related to the one of $F_{k=1,N}$. This is expressed by the following theorem⁶⁰

Theorem 5. *For a sequence a_N and $b_N > 0$ of real numbers and for a fixed integer $k > 1$, $F_{k,N}(a_N + b_N x)$ converges when $N \rightarrow \infty$ toward a non-degenerate cumulative probability $H_k(x)$ if and only if $F_{1,N}(a_N + b_N x)$ converges weakly to a non-degenerate cumulative probability $H(x)$, being one of the possible functions given by Theorem 3. If $H_k(x)$ exists then it is related to $H(x)$ through*

$$H_k(x) = H(x) \sum_{j=0}^{k-1} \frac{1}{j!} \left(\ln \frac{1}{H(x)} \right)^j. \quad (38)$$

In terms of probability density functions $h_k(x) = dH_k/dx$, the relation between the k^{th} largest and the largest value reads

$$h_k(x) = \frac{h(x)}{(k-1)!} \left(\ln \frac{1}{H(x)} \right)^{k-1}. \quad (39)$$

For instance, for random variables belonging to the Gumbel class, we have $H(x) = H_g(x)$ and

$$h^{(k)}(x) = \frac{1}{(k-1)!} \exp(-kx - e^{-x}). \quad (40)$$

In order to compare experimental or numerical data with theoretical predictions, it is convenient to make a slightly different choice for the parameters a_N and b_N , in such a way that some specific moments of the distribution are normalized to zero or one. This leads us to what we shall call in this paper the “standard forms” of the generalized extreme value distributions. Namely, one finds for the generalized Gumbel distribution with parameter k

$$g_k(x) = \frac{k^k \theta_k}{\Gamma(k)} \exp \left[-k\theta_k(x + \nu_k) - ke^{-\theta_k(x + \nu_k)} \right], \quad (41)$$

$$\theta_k^2 = \Psi'(k), \quad \nu_k = \frac{1}{\theta_k} [\ln k - \Psi(k)],$$

where $\Psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the digamma function. The generalized Gumbel distribution is illustrated on Fig. 1 for different values of k . Note that one can show that when k goes to infinity, the distribution $g_k(x)$ converges to the Gaussian distribution with zero mean and unit variance, while it converges to an exponential distribution for $k \rightarrow 0$.¹⁵⁸

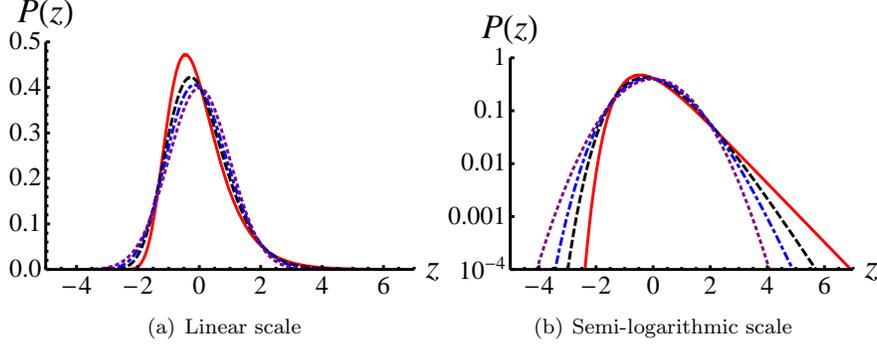


Fig. 1. Gumbel distribution $g_k(x)$ for $k = 1$ (plain), 3 (dash) and 10 (dot dash). The distributions are normalised to have a zero mean and a variance unity. Note the convergence to the Gaussian distribution (dot) when k increases.

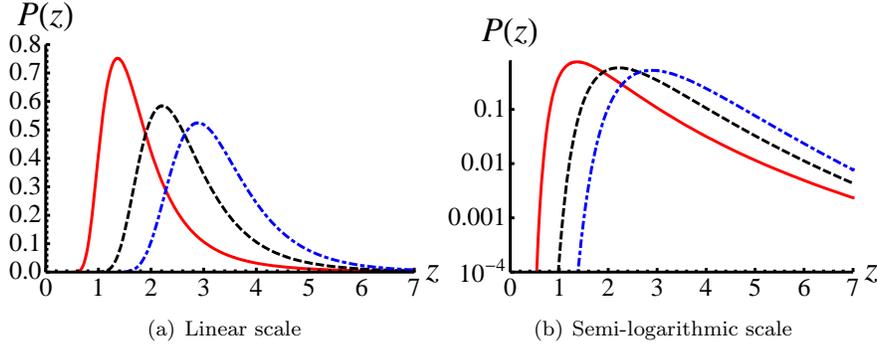


Fig. 2. Fréchet distribution $f_{k,\mu}(x)$ for $k = 2, 3$ and 4, with $\mu = 2$. Distributions are normalised to have a variance unity.

The generalized Fréchet distribution is given for $x > 0$ by

$$f_{k,\mu}(x) = \frac{\mu \lambda_f^k}{\Gamma(k)} \frac{1}{x^{1+k\mu}} \exp(-\lambda_f x^{-\mu}), \quad (42)$$

$$\lambda_f = \Gamma(k)^\mu \left[\Gamma(k) \Gamma\left(k - \frac{2}{\mu}\right) - \Gamma\left(k - \frac{1}{\mu}\right)^2 \right]^{-\mu/2}.$$

and by $f_{k,\mu}(x) = 0$ for $x \leq 0$. The generalized Weibull distribution reads for $x < 0$

$$w_{k,\mu}(x) = \frac{\mu \lambda_w^k}{\Gamma(k)} (-x)^{\mu k - 1} \exp(-\lambda_w (-x)^\mu), \quad (43)$$

$$\lambda_w = \Gamma(k)^{-\mu} \left[\Gamma(k) \Gamma\left(k + \frac{2}{\mu}\right) - \Gamma\left(k + \frac{1}{\mu}\right)^2 \right]^{\mu/2}.$$

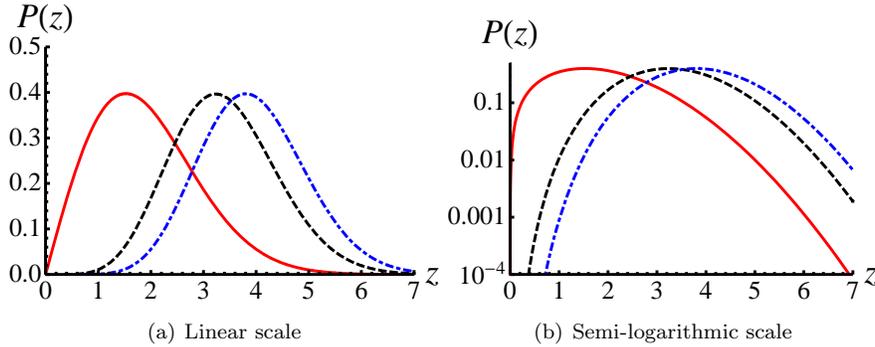


Fig. 3. Weibull distribution $w_{k,\mu}(x)$ for $k = 1, 3$ and 4 , with $\mu = 2$. Distributions are normalised to have a variance unity.

and $w_{k,\mu}(x) = 0$ for $x \geq 0$. Here the Fréchet and Weibull distributions are normalized such that their variance is normalized to one. For the Fréchet distribution, this is possible only when the first moment is finite, that is for $k > 1/\mu$. The Fréchet and Weibull distributions are shown on Fig. 2 and Fig. 3 respectively.

In the context of extreme value statistics, $\Gamma(k) = (k-1)!$ as the parameter k is by definition an integer. Still, once written using Gamma functions, these distributions can formally be extended to positive real values of k , although one then loses a direct interpretation in terms of extreme value statistics. We shall come back to the interpretation of these distributions with noninteger values of k in section 4.3.3. To conclude this discussion let us mention that some results exist concerning the speed of convergence towards asymptotic extreme value distributions,^{60,64,66} but such results go beyond the scope of the present review.

3.3. Broad distributions: when extreme values change the statistics of sums

3.3.1. A simple scaling argument on the largest term in the sum

As mentioned in Section 3.1.1, the standard Central Limit Theorem, associated to a Gaussian asymptotic distribution, breaks down as soon as the second moment of the individual variables x_i is divergent, i.e., formally infinite (note however the slight extension allowed by Theorem 2).

Generically, distributions $p(x)$ leading to such a divergence of $\langle x_i^2 \rangle$ are characterized by a power-law decay at large argument (we assume here for simplicity that $x > 0$)

$$p(x) \sim \frac{c}{x^{1+\alpha}}, \quad x \rightarrow \infty \quad (\alpha > 0). \quad (44)$$

In this case, $x^q p(x)$ behaves at large x as $1/x^{1+\alpha-q}$, so that the q^{th} moment $\langle x^q \rangle$ converges only if $q < \alpha$. Accordingly, the second moment $\langle x^2 \rangle$ is infinite as soon as $\alpha \leq 2$, and the first moment $\langle x \rangle$ is infinite if $\alpha \leq 1$.

This last case is indeed very instructive in order to get an intuitive feeling of the somehow surprising behavior of sums of broadly distributed variables. As mentioned in the introduction, such sums are often dominated by the largest terms. To see qualitatively how this happens, let us estimate the largest value among the set $\{x_i\}$. Introducing $\tilde{F}(z) \equiv \int_z^\infty p(x)dx$, one has from Eq. (28)

$$F_N^{\max}(z) \equiv \text{Prob}(\max(x_1, \dots, x_N) < z) = [1 - \tilde{F}(z)]^N \quad (45)$$

For large N , relevant values of z are also typically large, so that $\tilde{F}(z)$ is small, and can be approximated as $\tilde{F}(z) \approx c'z^{-\alpha}$, with $c' = c/\alpha$. One then obtains the asymptotic large N expression:

$$F_N^{\max}(z) \approx e^{-N\tilde{F}(z)} \approx e^{-c'Nz^{-\alpha}}. \quad (46)$$

Accordingly, the cumulative distribution $F_N^{\max}(z)$ can be written as a function of the rescaled variables $z/N^{1/\alpha}$, which shows that typical values of the maximum of the x_i 's are of the order of $N^{1/\alpha}$.

Coming back to the sum $\sum_{i=1}^N x_i$, one expects from the laws of large numbers that the typical value of the sum is proportional to N at large N , namely $\sum_{i=1}^N x_i \approx N\langle x \rangle$. This should be true at least when $\langle x \rangle$ is finite. Indeed, if $\alpha > 1$, the largest term in the sum is of the order of $N^{1/\alpha}$, and becomes negligible with respect to N at large N . Hence the largest term does not dominate the sum. In contrast, if $\alpha < 1$, $N^{1/\alpha} \gg N$ at large N , so that the sum can no longer be proportional to N (which is consistent with the divergence of $\langle x \rangle$). In this case, the largest term dominates the sum, and the latter scales, similarly to the largest term, as $N^{1/\alpha}$. Note however that the total contribution of the other terms in the sum does not become negligible even when $N \rightarrow \infty$. Only the scaling behavior of the sum with N is the same as that of the largest terms, but the total contribution of the other terms is itself of the order of $N^{1/\alpha}$. This explains why, although sums of broadly distributed random variables are dominated by extreme values (the largest terms), the distribution of such sums does not belong to the classes of asymptotic extreme value distributions. Note however that both the Lévy-stable distributions (see Sect. 3.3.2 below) and the Fréchet distribution share a power-law tail with the same exponent.

3.3.2. Generalized Central Limit Theorem and Lévy-stable laws

In the case when the second moment $\langle x_i^2 \rangle$ is infinite, the standard Central Limit Theorem no longer applies (apart from the slight extension given in Theorem 2). Yet, it is still possible to find rescaling parameters a_N and b_N such that the distribution of the rescaled random variable z_N ,

$$z_N = \frac{1}{b_N} \left(\sum_{i=1}^N x_i - a_N \right) \quad (47)$$

converges to a limit distribution when $N \rightarrow \infty$, which belongs to a family of distributions called Lévy-stable laws. These stable distributions depend on two shape

parameters α and β , with $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$, and two scales parameters. This means that keeping the shape of the distribution constant, one can always build another stable distribution through a translation and dilatation of the variable, as in the case of extreme value statistics. Since these scale parameters can be eliminated through a redefinition of a_N and b_N , we shall not make explicit reference to them in the following. Hence, we shall make a standard choice of scale parameters, and simply denote the Lévy-stable laws as $L(z; \alpha, \beta)$. The following generalized central limit theorem for sums of broadly distributed variables then holds.^{54,55,56,5}

Theorem 6. *Let $\{x_k\}_{k=1,\dots,N}$ be a sequence of i.i.d. random variables with cumulative probability distribution $F(x)$. We define the random variable $y_N = \sum_{k=1}^N x_k$. Let α and β be two real numbers such that $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$. Then $F(x)$ belongs to the attraction basin of the Lévy distribution $L(z; \alpha, \beta)$ defined by its characteristic function*

$$\hat{L}(k; \alpha, \beta) = \exp[-|k|^\alpha (1 + i\beta \operatorname{sgn}(k)\varphi(k, \alpha))], \quad (48)$$

$$\varphi(k, \alpha) = \tan \frac{\pi\alpha}{2} \quad \text{if } \alpha \neq 1, \quad (49)$$

$$\varphi(k, \alpha) = \frac{2}{\pi} \ln |k| \quad \text{if } \alpha = 1, \quad (50)$$

if the following conditions hold:

$$\lim_{x \rightarrow \infty} \frac{F(-x)}{1 - F(x)} = \frac{1 - \beta}{1 + \beta}, \quad (51)$$

$$\lim_{x \rightarrow \infty} \frac{1 - F(x) + F(-x)}{1 - F(rx) + F(-rx)} = r^\alpha, \quad \forall r > 0. \quad (52)$$

The scale parameters are chosen such that the Fourier transform $\hat{L}(k; \alpha, \beta)$ of $L(z; \alpha, \beta)$ has the simple form (48). Apart from a few specific values of α and β , no explicit expression for $L(z; \alpha, \beta)$ is known in general, but asymptotic expressions (for instance, at large $|z|$) can be derived from the Fourier transform. For $\alpha = 2$, $\varphi(k, 2) = 0$, and

$$\hat{L}(k; 2, \beta) = \exp(-k^2) \quad (53)$$

for all values of β , so that $L(z; 2, \beta)$ is simply a Gaussian distribution. For $\alpha < 2$, $L(z; \alpha, \beta)$ depends on β , which characterizes the asymmetry of the distribution; the case $\beta = 0$ then corresponds to a symmetric limit distribution. Another case where the distribution is known explicitly is when $\alpha = 1$ and $\beta = 0$, corresponding to the Cauchy distribution:

$$L(z; 1, 0) = \frac{1}{\pi(1 + z^2)}. \quad (54)$$

In practice, the parameters α and β can be obtained from the distribution $p(x) = dF/dx$ in the following way. If $p(x)$ behaves asymptotically as

$$p(x) \sim \frac{c_+}{x^{1+\alpha_+}}, \quad x \rightarrow +\infty, \quad (55)$$

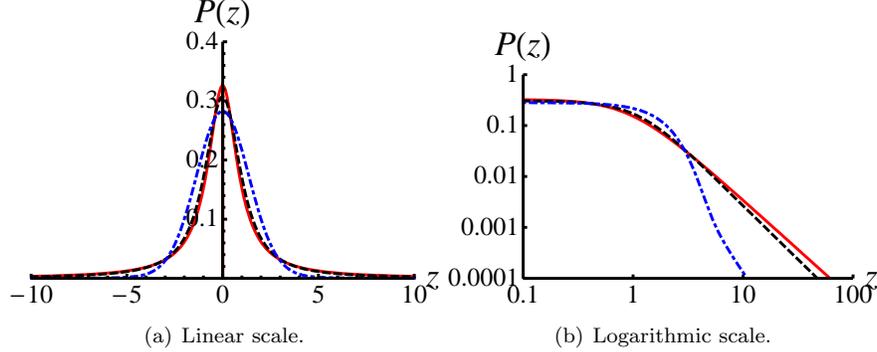


Fig. 4. Examples of Lévy distributions, for $\beta = 0$ and $\alpha = 0.95$ (plain), $\alpha = 1.1$ (dash) and $\alpha = 1.9$ (dot dash).

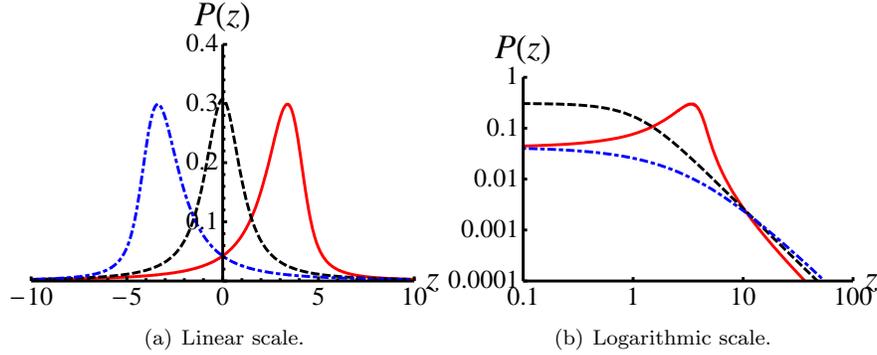


Fig. 5. Examples of Lévy distributions, for $\alpha = 1.1$ and $\beta = -0.5$ (plain), $\beta = 0$ (dash) and $\beta = 0.5$ (dot dash). Distributions have been centered such that $\langle z \rangle = 0$.

$$p(x) \sim \frac{c_-}{|x|^{1+\alpha_-}}, \quad x \rightarrow -\infty, \quad (56)$$

with $0 < \alpha_+, \alpha_- \leq 2$, then from Eqs. (51) and (52), α and β are given by

$$\alpha = \min(\alpha_+, \alpha_-), \quad (57)$$

$$\beta = \frac{c_+ - c_-}{c_+ + c_-} \quad \text{if } \alpha_+ = \alpha_-, \quad (58)$$

whereas $\beta = 1$ if $\alpha_+ < \alpha_-$ and $\beta = -1$ if $\alpha_+ > \alpha_-$. When $\beta = 1$, the distribution $L(z; \alpha, \beta)$ vanishes for $z < 0$; similarly, it vanishes for $z > 0$ if $\beta = -1$. This means that for instance a broad negative tail with exponent α_- actually “disappears” from the limit distribution if the positive tail is even broader, that is if $\alpha_+ < \alpha_-$. Note that when $L(z; \alpha, \beta)$ is nonzero on a given domain ($z < 0$ or $z > 0$), its corresponding tail has the same exponent as the original distribution $p(x)$ on the same domain. For completeness, let us also mention the behavior with N of the

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rescaling coefficients a_N and b_N . The coefficient b_N is of the form $b_N = b_0 N^{1/\alpha}$, while $a_N = N\langle x \rangle + a_0$ if $1 < \alpha < 2$ and $a_N = 0$ if $0 < \alpha < 1$, where a_0 and b_0 are some constants. The case $\alpha = 1$ involves logarithmic corrections.

3.3.3. Example of application: laser cooling of atoms

Although many systems obey the standard CLT, numerous different examples of broad distribution effects in physical systems have been reported in the last decades. To give only a few examples, these range from glassy dynamics,^{4,6,74,75} anomalous diffusion in disordered media,^{5,76,77,78,79} diffusion of spectral lines in disordered solids,^{69,70} and random walks in solutions of micelles,⁷¹ to laser cooling of atomic gases,^{8,9} laser trapped ions,^{72,73} fluorescence of single nanocrystals,¹³ or chaotic transport^{10,11} and turbulent flows¹².

As an illustration, let us consider the case of laser cooling experiments, which consist in reducing the dispersion of momenta of atoms through interaction of the atoms with photons. One of such cooling protocols is called subrecoil laser cooling, and it consists in reducing the momentum spread of the atoms to a value much smaller than the typical momentum $\hbar k$ of the photons.^{8,9,80} Denoting by q the magnitude of the atomic momentum \mathbf{q} , the atom-photon interaction process is such that the scattering rate of the photon $R(q)$ vanishes for $q = 0$. In most cases, $R(q)$ behaves in the vicinity of $q = 0$ as a power law of q with an even exponent: $R(q) \sim q^\beta$, $q \rightarrow 0$, with $\beta = 2, 4$, or 6 , depending on the specific atom-photon interaction process. This means that the average time spent at momentum q between two photon scatterings, to be denoted as the lifetime $\tau(q)$, behaves at small q as a diverging power law of q :

$$\tau(q) = \tau_0 \left(\frac{q_m}{q} \right)^\beta \quad (59)$$

where τ_0 and q_m are time and momentum scales respectively (note that $q_m \ll \hbar k$). After a photon scattering, the atom typically has a momentum $q \approx \hbar k$, so that the scattering rate becomes high. Then successive scatterings will rapidly make the atomic momentum come back to a value $q < q_m$, where it will stay again for a long time. Neglecting the time spent in the region $q > q_m$, and assuming that the values of the momentum \mathbf{q} are uniformly distributed in the D -dimensional sphere of radius q_m , the distribution of q reads:

$$\rho(q) = D \frac{q^{D-1}}{q_m^D}, \quad 0 < q < q_m. \quad (60)$$

Let us now determine the distribution $\psi(\tau)$ of the lifetime τ . From the relation $\psi(\tau)|d\tau| = \rho(q)|dq|$, one finds

$$\psi(\tau) = \frac{c}{\tau^{1+\alpha}}, \quad \tau > \tau_{\min}, \quad (61)$$

where c and τ_{\min} are parameters that can be expressed as a function of τ_0 , q_m and D . The exponent α is given by $\alpha = D/\beta$. Note that τ is the average time

spent at momentum q , and that the actual time τ_s spent at momentum q fluctuates according to an exponential distribution of mean τ . Despite these fluctuations, the distribution of τ_s averaged over q behaves similarly to $\psi(\tau)$ at large τ , so that we shall not distinguish between the two notions in what follows.

The time T_N of the N^{th} scattering can be expressed as $T_N = \sum_{i=1}^N \tau_i$. If $\alpha < 2$, the fluctuations of T_N at large N are no longer Gaussian, but distributed according to the Lévy-stable distribution of index α , and asymmetry parameter $\beta = 1$. If $\alpha < 1$, the typical value of T_N is no longer proportional to N , but rather to $N^{1/\alpha}$, since the formal average value of τ is infinite. This has important consequences on the behavior of the system. Assuming that a stationary state is reached, the probability density $p_{\text{st}}(q)$ to find an atom at momentum q is proportional to the density $\rho(q)$, and to the lifetime $\tau(q)$:

$$p_{\text{st}}(q) = \frac{1}{\langle \tau \rangle} \rho(q) \tau(q), \quad (62)$$

with $\langle \tau \rangle = \int_0^{q_m} dq \rho(q) \tau(q)$. For $\alpha < 1$, the distribution $p_{\text{st}}(q)$ is no longer normalizable since $\langle \tau \rangle$ is infinite, so that no stationary distribution can be reached. This means that the dynamical distribution of q becomes more and more peaked around $q = 0$ as time evolves, allowing atoms to reach lower and lower momenta, or equivalently, temperatures.

The same behavior can also be found in other physical contexts, like the aging phenomenon in glassy systems. Actually, the kinetic mechanism at play in the trap model of aging dynamics is precisely the same as for laser cooling, although its physical origin is different. Both models can actually be recast into the same formalism once expressed in terms of lifetimes τ only.⁸⁰

3.4. Convergence to the Gaussian distribution: 'finite-size' effects

3.4.1. Finite N deviations from the Gaussian distribution

In the previous section we presented mathematical results concerning the sum of random variables. Obviously, these results rely on hypotheses that may or may not be satisfied in physical systems. Note also that the observation of Gaussian distributions in physical systems does not mean that the CLT applies, that is to say that the local variables are i.i.d., and that the second moment is finite: Theorem 1 is only an implication, not an equivalence. Theorem 2 is an equivalence between the identity (15) and the Gaussian distribution, only assuming that the variables are i.i.d.

In most physical systems it is hard to know a priori whether these hypotheses on the microscopic variables are valid or not. Indeed, one most often studies fluctuations of macroscopic quantities to gain a better understanding on the statistical properties at the microscopic level. In this respect, the observation of *non*-Gaussian fluctuations is much more interesting from the physicist's point of view: from the contra-positive of the CLT, either the hypotheses are *not* true for the physical sys-

tem considered, or the number of microscopic random variables is not large enough to ensure the convergence towards the limit distribution. Mathematics therefore makes observation of non-Gaussian fluctuations much more interesting: it is generally the signature of non-trivial physical phenomena in the system, leading either to correlations or to broad distributions of the microscopic degrees of freedom.

However, before drawing such conclusions, one has to check that the number of individual random variables is indeed large enough to use a limit theorem. At first sight this condition could seem rather easy to fulfill, but things might sometimes be more subtle than expected, as we shall illustrate below. Let us imagine that we know that the considered quantity is a sum of i.i.d. random variables, each with a finite variance. Mathematics and the CLT then tell us that the *limit* distribution of our quantity is a Gaussian distribution. This is a strong result, with a clear-cut assumption: the variance is defined or not. The problem is that the CLT does not in itself specify how fast the convergence is, so that the number of (random) terms needed to actually observe the convergence might be too large to be physically accessible. Characterizing the convergence of the distribution of random sums towards the normal distribution is thus an important issue. This is the aim of the following theorem.^{57,58,81}

Theorem 7. *Let $\{x_i\}_{i=1..N}$ be a set of N random variables, independent and identically distributed according to the distribution $p(x)$, with $\langle x \rangle = m$, $\langle (x - m)^2 \rangle = \sigma^2$, $0 < \sigma < \infty$, and $\langle |x - m|^3 \rangle = \rho \sigma^3 < \infty$. Then, for any integer $N > 0$, the cumulative distribution function $F_N(y)$ of the variable*

$$y = \frac{1}{\sigma\sqrt{N}} \left(\sum_{i=1}^N x_i - Nm \right) \quad (63)$$

satisfies the following inequality, called the Berry-Esseen inequality:

$$\sup_y \left| F_N(y) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx \right| \leq \frac{A\rho}{\sqrt{N}} \quad (64)$$

where A is an absolute positive constant, independent of N and of the distribution $p(x)$.

Hence we see that the speed of convergence to the normal law is quite slow, like $1/\sqrt{N}$. In addition, the departure from the cumulative normal distribution depends on the details of the distribution $p(x)$, through the ratio ρ of the third centered moment to the variance to the power $3/2$. If ρ is large, the convergence to the normal law can be so slow that it may not be achieved on physically accessible scales, as illustrated on the following example.

3.4.2. *Broad distribution effects in laws with finite but large variance*

The lack of convergence to the normal law on experimentally accessible scales has been observed by Da Costa and coworkers in experiments on tunnel junctions.⁸²

These authors studied the tunnel current measured through metal-oxide junctions, in particular when the metal is Cobalt. Their results show a strong heterogeneity of the local currents at the film surface. The probability distribution of the local current i_{loc} can be determined experimentally and turns out to be non-Gaussian, and in good agreement with a lognormal distribution, as predicted earlier by Bardou.⁸³

$$p(i_{\text{loc}}) = \frac{1}{\beta i_{\text{loc}} \sqrt{2\pi}} \exp \left[-\frac{1}{2\beta^2} \left(\ln \frac{i_{\text{loc}}}{i_0} \right)^2 \right], \quad (65)$$

where β and i_0 are two junction-dependent parameters. This lognormal distribution can rather easily be understood in terms of Gaussian fluctuations of the height of the junction surface.^{83,82}

Now one can ask what is the distribution of the total current through the junction. Decomposing the area of the junction into small regions, this current is the sum of local currents, assumed to be independent: $I = \sum_{k=1}^N i_{\text{loc},k}$. The total current is therefore the sum of i.i.d. random variables, and as the moments of the lognormal distribution are finite, the Central Limit Theorem applies: we thus expect Gaussian fluctuations for the total current. Experimental results however exhibit strong deviations from the Gaussian distribution: measured distributions are indeed broad and skewed, even if the estimated value of N is rather large. The reason for this observation is found in the very slow convergence toward the Gaussian distribution, which turns out to be out of reach in this experimental system.⁸⁴ To check this point, we estimate the value of the parameter ρ in Theorem 7, which requires to evaluate first the moments. From Eq. (65), the moments of the lognormal distribution are given by

$$\langle i_{\text{loc}}^n \rangle = i_0^n e^{\beta^2 n^2 / 2} \quad (66)$$

Using the inequality $\langle |x - \langle x \rangle|^3 \rangle \geq \langle (x - \langle x \rangle)^3 \rangle$, one finds

$$\rho \geq \frac{\langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle + 2\langle x \rangle^3}{(\langle x^2 \rangle - \langle x \rangle^2)^{3/2}}. \quad (67)$$

Together with Eq. (66), this last inequality leads to

$$\rho \geq \frac{e^{3\beta^2} - 3e^{\beta^2} + 2}{(e^{\beta^2} - 1)^{3/2}} \quad (68)$$

With the experimental value $\beta = 1.84$, one obtains the estimate $\rho \gtrsim 170$. Assuming that the constant A is of the order of 1 (it can be shown⁵⁷ that $A \geq 1/\sqrt{2\pi}$), one sees that Theorem 7 does not impose any constraint on the convergence to the normal law until N reaches values of the order of $\rho^2 \approx 3 \times 10^4$. Let us also note that the typical value of the current, that is the locus of the maximal probability, is

$$i_{\text{typ}} = i_0 e^{-\beta^2}, \quad (69)$$

yielding a ratio

$$\frac{\langle i_{\text{loc}} \rangle}{i_{\text{typ}}} = e^{\frac{3}{2}\beta^2} \approx 160. \quad (70)$$

So the average value is much larger than the typical value, which is a standard property of broad distributions with finite first moment. The typical value is experimentally relevant as it corresponds to the most probable value measured in an experiment.

This example shows that one should keep in mind that observing non-Gaussian fluctuations might be the result of a lack of convergence toward the normal distribution, due to effective broad distribution effects appearing in finite-size systems. The lognormal distribution precisely exhibits such properties if the parameter β is large enough (typically $\beta \gtrsim 1.5$). Similar effects also appear in truncated power-law distributions, namely distributions $p(x)$ such that $p(x) \propto x^{-1-\alpha}$ for $X_{\min} < x < X_{\max}$ and with an exponential decay for $x > X_{\max}$ (hence all moments are finite). If $0 < \alpha < 2$ and $X_{\min} \ll X_{\max}$, one typically has $\rho \sim (X_{\max}/X_{\min})^{\alpha/2}$, so that the convergence to the normal law is generically very slow.

4. Sums and extreme values of dependent or non-identically distributed variables

4.1. *Statistics of random sums*

In this section we shall discuss the case of dependent or non identically distributed random variables. These cases are very interesting from the physicist viewpoint, and of course turn out to be quite subtle on the mathematical side, apart from some simple cases. We shall first explore what happens when the limits of validity of Central Limit Theorem are reached. While from a mathematical point of view, the hypotheses underlying the CLT are either true or not, the boundaries are actually blurred in physics. But after all, it is part of the physicist's *savoir-faire* to play with these blurred boundaries.

4.1.1. *Extension of the CLT to non-identically distributed variables*

Up to now we only considered the case of independent and identically distributed random variables. Before including correlations, one can consider the case when the variables are still independent, but no longer identically distributed. In other words the joint probability of N random variables still factorizes, but the product involves different marginal distributions:

$$J_N(x_1, \dots, x_N) = \prod_{i=1}^N p_i(x_i). \quad (71)$$

The factorization property makes this case analytically tractable, and some mathematical results are thus available. It turns out that a slightly generalized form of the CLT can be applied, provided that the so-called Lindeberg condition⁵³ is fulfilled.

Theorem 8. *Let $\{x_k\}_{k=1, \dots, N}$ be a sequence of independent random variables with finite first and second moments and let $p_k(x_k)$ be the probability distribution of x_k .*

In order that the probability distribution of the normalized sum

$$S_N = \frac{1}{b_N} \left(\sum_{k=1}^N x_k - a_N \right), \quad (72)$$

with

$$a_N = \sum_{k=1}^N \langle x_k \rangle, \quad b_N = \left(\sum_{k=1}^N (\langle x_k^2 \rangle - \langle x_k \rangle^2) \right)^{\frac{1}{2}}, \quad (73)$$

converges to the normal law, it is necessary and sufficient that the Lindeberg condition holds for all $\epsilon > 0$, namely

$$\lim_{N \rightarrow \infty} \frac{1}{b_N^2} \sum_{k=1}^N \int_{|v| > \epsilon b_N} dv v^2 p_k(v + \langle x_k \rangle) = 0. \quad (74)$$

An intuitive (although not fully correct) interpretation of the Lindeberg condition is that the summands become infinitesimal with respect to the sum in the large N limit. The Lindeberg condition is straightforwardly satisfied in the case of i.i.d. random variables. In this case, the condition reads

$$\lim_{N \rightarrow \infty} \frac{1}{b_1^2} \int_{|v| > \epsilon b_N} dv v^2 p(v + \langle x \rangle) = 0. \quad (75)$$

As b_N goes to infinity when $N \rightarrow \infty$, condition (75) is obviously satisfied.

Note that the Lindeberg condition implies that the standard deviation b_N of the sum (or equivalently the variance b_N^2) diverges with N . To show this, it is sufficient to show that the Lindeberg condition cannot be fulfilled if b_N is bounded. In this case, since b_N increases with N , it necessarily converges to a finite limit. Hence the terms under the sum in Eq. (74) asymptotically become independent of N . As all terms are strictly positive (at least for small enough ϵ), the sum is larger than a positive bound, and the limit in Eq. (74) cannot be zero, since the prefactor $1/b_N^2$ also converges to a finite limit.

4.1.2. $1/f$ -noise: an example of non-identically distributed variables

If the Lindeberg condition is not satisfied, one could observe other distributions depending on the particular problem: there is no general or universal behavior and one has to study each case separately. A simple and explicit example of such problems in physics was given by Antal and coworkers.⁹² These authors proposed a simple model for $1/f$ -noise, that we recast here in a slightly different form. In this model, one considers a one-dimensional random signal h_ℓ parameterized by an integer $\ell = 0, \dots, N-1$. This discretized signal may be interpreted either as a time signal or as a spatial signal in a one-dimensional system. Decomposing the signal

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h_ℓ using a discrete Fourier transform, the complex Fourier amplitude c_n associated with the wavenumber $q_n = 2\pi n/N$, $n = 0, \dots, N-1$ reads:

$$c_n = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} h_\ell e^{-iq_n \ell}. \quad (76)$$

The basic assumption of this simple model of $1/f$ -noise is that the Fourier coefficients c_n , $n > 0$, are independent complex random variables with Gaussian distribution and variance proportional to $1/n$:

$$p_n(c_n) = \frac{n\kappa}{\pi} e^{-n\kappa|c_n|^2}. \quad (77)$$

The value of c_0 is unimportant to the following, since we shall consider fluctuations of h_ℓ around the empirical mean $\bar{h} = N^{-1} \sum_{\ell=0}^{N-1} h_\ell = c_0/\sqrt{N}$. Hence we define the 'roughness' of the signal as its empirical variance, namely

$$E_N = \sum_{\ell=0}^{N-1} (h_\ell - \bar{h})^2, \quad (78)$$

This quantity, which fluctuates from one realization of the signal to another, is a natural global observable in this context. From Parseval's theorem, it may be conveniently reformulated as a function of the Fourier amplitudes

$$E_N = \sum_{n=1}^{N-1} |c_n|^2, \quad (79)$$

so that we shall also call E_N the integrated power spectrum (or total energy) of the signal.

The integrated power spectrum E_N can thus be written as a sum of independent, but non-identically distributed variables $u_n \equiv |c_n|^2$. From the Gaussian distribution of c_n in the complex plane, it is straightforward to show that the distribution of u_n is given by

$$\tilde{p}_n(u_n) = n\kappa e^{-n\kappa u_n} \quad (80)$$

As a result, the variance of u_n is equal to κ^2/n^2 , so that $\text{Var}(E_N) = \sum_{n=1}^{N-1} \text{Var}(u_n)$ converges to a finite limit as $N \rightarrow \infty$; accordingly, Lindeberg's condition does not hold.

So as to deal with the infinite N limit, it is convenient to introduce a rescaled variable $\varepsilon = (E_N - \langle E_N \rangle)/\sigma$, where σ^2 is the infinite N limit of the variance of E_N , namely $\sigma^2 = \sum_{n=1}^{\infty} \kappa^2/n^2$. Denoting as $\psi_N(\varepsilon)$ the PDF of ε , we introduce the associated characteristic function $\chi_N(\omega)$ defined as

$$\chi_N(\omega) = \int_{-\infty}^{\infty} d\varepsilon \psi_N(\varepsilon) e^{-i\omega\varepsilon}. \quad (81)$$

Using the independence property of the variables y_n , one can express $\chi_N(\omega)$ as the product of the characteristic functions of u_n , $n = 1, \dots, N-1$. In the infinite

N limit, $\chi_N(\omega)$ converges to the asymptotic characteristic function $\chi_\infty(\omega)$ given by^{92,93}

$$\chi_\infty(\omega) = \prod_{n=1}^{\infty} \left(1 + \frac{i\omega}{n\sigma\kappa}\right)^{-1} \exp\left(\frac{i\omega}{n\sigma\kappa}\right). \quad (82)$$

This expression can be transformed using the following relation,⁹⁴ valid for any complex number $z \neq -1, -2, \dots$,

$$\Gamma(1+z) = e^{-\gamma z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + \frac{z}{n}}, \quad (83)$$

where $\Gamma(\cdot)$ is the usual Euler Gamma function, and $\gamma = 0.577\dots$ is the Euler constant. Computing the inverse Fourier transform of $\chi_\infty(\omega)$, one finds that the limiting distribution $\psi_\infty(\varepsilon)$ is precisely a Gumbel distribution

$$\psi_\infty(x) = g_1(x) = \exp[-(bx + \gamma) - e^{-(bx+\gamma)}], \quad b = \frac{\pi}{\sqrt{6}}, \quad (84)$$

as introduced in Eq. (41). This result is rather striking since there is a priori here no relation with extreme values statistics. Understanding whether the appearance of the Gumbel distribution in the present context is a coincidence, or if it unveils some deep connection between sums of random variables and extreme values is the motivation of several studies.^{40,144,146,96} We shall discuss this point in details in Sect. 4.3.

Finally, the example of the $1/f$ -noise model illustrates how correlated random variables (here the physical signal h_ℓ) may in some simple cases be converted into independent, but non-identically distributed variables (the amplitudes c_n) through a Fourier transform. Spatial correlations in the system then come from the fact that the Fourier modes are extended objects. The key point is that the global quantity of interest, namely the integrated power spectrum E can be expressed either as a function of the physical signal $E = \sum_\ell (h_\ell - \bar{h})^2$ or as a function of the Fourier amplitudes $E = \sum_n |c_n|^2$, thanks to Parseval's theorem. Hence the statistics of the power spectrum can equivalently be considered as a problem of sum of correlated variables or a problem of sum of independent, but non-identically distributed random variables. However, this is a rather specific class of problems, and in more general cases, correlated variables cannot be converted into independent variables. It is thus necessary to develop different approaches to tackle this issue.

4.1.3. Correlated and identically distributed variables: scaling arguments

Generically, the case of correlated random variables is very difficult from a mathematical point of view. To the best of our knowledge, there is no general criterion as Lindeberg's condition, to ensure the applicability of the CLT for generic correlated variables. However, some rigorous results exist in some specific cases. For instance, it is known that the CLT still holds for a particular case of correlation,

the martingale differences.^{97,98,56} Convergence theorems for non-linear functionals of stationary Gaussian sequences with power-law correlation have also been obtained.^{99,100,102,101,103,104,105} We shall come back to this issue in Sect. 4.1.4. In addition, propositions have been made recently concerning extensions of the CLT to specific classes of correlated variables using mathematical concepts like deformed products,¹⁰⁶ or in relation with Tsallis' non-extensive entropy.^{107,108,109} However, this area is still a matter of debate.^{111,112,113,114}

From a less rigorous point of view, different strategies may be used, from simple scaling arguments to more involved renormalization group approaches (see Sect. 4.1.5). Let us start with a simple and intuitive physical argument. Considering a large system of linear size L in dimension D , we assume that the microscopic degrees of freedom are typically correlated over a length $\xi < L$. Let us now imagine that we are interested in the statistics of a particular global observable. The latter can be expressed as a sum of local quantities computed on subsystems with a linear size of the order of ξ . Then as a first approximation, the quantities computed on two different subsystems are statistically independent, so that the global observable can be estimated as a sum of $N = (L/\xi)^D$ i.i.d. random variables. The main issue is now the behavior of the correlation length ξ with the system size L . If $L/\xi \rightarrow \infty$ when taking the thermodynamic limit $L \rightarrow \infty$, then the number N of independent terms in the sum goes to infinity. At a heuristic level, one can apply the central limit theorem, leading to a Gaussian distribution for the sum. In contrast, if the correlation length scales with the system size, that is $\xi \sim L$, then N remains finite and the central limit theorem does not apply. In this case, the distribution obtained in the limit $L \rightarrow \infty$ is generically not a Gaussian distribution. Note that this heuristic argument is qualitatively consistent with the Lindeberg condition: if the effective number of degrees of freedom remains finite when $N \rightarrow \infty$, one expects that the variance of the sum also converges, and Lindeberg's condition does not hold.

It is possible to make the above scaling argument sharper (though not fully rigorous) using thermodynamic concepts as we shall now illustrate in a “magnetic language” for definiteness –although nothing is specific to magnetic systems here. Consider a spin model with spins $\{s_i\}$, $i = 1, \dots, N$ interacting through a Hamiltonian $\mathcal{H}_0(\{s_i\})$. In the presence of an external magnetic field h , the Hamiltonian becomes $\mathcal{H}(\{s_i\}) = \mathcal{H}_0(\{s_i\}) - h \sum_{i=1}^N s_i$. The partition function Z is given by:

$$Z = \sum_{\{s_i\}} e^{-\beta \mathcal{H}(\{s_i\})}. \quad (85)$$

Then the successive derivatives of the free energy $F(\beta, h) = -(1/\beta) \ln Z$ yield the cumulants of the total magnetization $M = \sum_{i=1}^N s_i$:

$$\langle M^k \rangle_c = \frac{\partial^k F}{\partial h^k} = N \frac{\partial^k f}{\partial h^k}, \quad (86)$$

where $f(\beta, h) = F(\beta, h)/N$ is the free energy per spin. Since both $f(\beta, h)$ and h are intensive quantities, it follows that all the cumulants are proportional to N . In

particular, the average value $\langle M \rangle$ and the variance σ_N^2 of M scale with N . Let us thus write the variance as $\sigma_N^2 = N\sigma^2$. To check whether an asymptotic distribution exists, one needs to consider the reduced variable z defined as

$$z = \frac{M - \langle M \rangle}{\sigma\sqrt{N}}. \quad (87)$$

The average value of z vanishes by definition of z , and the cumulants of order $k \geq 2$ are actually not affected by the shift by $\langle M \rangle$, but only by the rescaling factor $1/\sigma\sqrt{N}$. Accordingly, these cumulants are given by

$$\langle z^k \rangle_c = \frac{N}{\sigma^k N^{k/2}} \frac{\partial^k f}{\partial h^k}, \quad k \geq 2. \quad (88)$$

Hence the cumulant of order $k \geq 2$ is proportional to $N^{1-k/2}$: the second order cumulant remains finite when N goes to infinity, while higher order cumulants vanish in this limit. This precisely means that the distribution of z becomes Gaussian in the thermodynamic limit.

Note however that the above result implicitly assumes that all partial derivatives of $f(\beta, h)$ with respect to h are finite, i.e., that $f(\beta, h)$ is regular. Knowing whether the free energy per degree of freedom is well-defined and regular in the thermodynamic limit is a difficult mathematical problem, related in particular to the theory of large deviations.¹¹⁵ From a physicist viewpoint, it is well-known that at a second order critical point where correlations become strong, the second order derivative of the free energy with respect to h (namely the susceptibility) diverges, and the second order cumulant of M no longer scales with N . Thus the above thermodynamic argument breaks down, and the asymptotic distribution can be distinct from a Gaussian, meaning that the effective number of degrees of freedom remains finite. This is due to the fact that the two-point correlation function behaves as a power law, and that the only length scale in the problem is the system size (apart from the microscopic length scale), so that $\xi \sim L$. Therefore, from a more general point of view, deviations from the Gaussian distribution may be expected when the two-point correlation of a random sequence behaves as a power law.

4.1.4. Taqg's reduction theorem for Gaussian stationary sequences

General theorems about correlated variables are seemingly difficult to obtain. However, interesting limit theorems have been obtained for special classes of correlated variables, namely non-linear functionals of stationary Gaussian sequences.^{99,100,102,101,103,104,105} A Gaussian sequence is characterized by a Gaussian joint probability density¹¹⁶ (we consider here the case $\langle x_i \rangle = 0$ for simplicity)

$$J_N(x_1, \dots, x_N) = \frac{\sqrt{\det R}}{(2\pi)^{N/2}} \exp\left(-\frac{1}{2} \sum_{i,j=1}^N x_i R_{ij}^{-1} x_j\right), \quad (89)$$

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where R is a matrix of elements R_{ij} , and R_{ij}^{-1} are the matrix elements of the inverse matrix R^{-1} . If the matrix elements are such that $R_{ij} = r(|i - j|)$ where $r(m)$ is a given function, then the Gaussian sequence is stationary.^c In this case, the marginal probability for each random variable is a normal distribution of variance $r(0)$, and the two-point correlation function reads $\langle x_i x_{i+m} \rangle = r(m)$.

In the following, we focus on Gaussian stationary sequences x_1, \dots, x_N such that the correlation function $r(m)$ decays at large distance as a power law, $r(m) \sim m^{-\alpha}$ (note that for a rigorous definition of this power law behavior, the limit $N \rightarrow \infty$ should be taken before the limit $m \rightarrow \infty$). We then introduce a sequence of variables y_1, \dots, y_N through

$$y_i = \psi(x_i), \quad i = 1, \dots, N, \quad (92)$$

where $\psi(x)$ is a regular function taking real values, and such that

$$\int_{-\infty}^{\infty} dx \psi(x) e^{-x^2/2} = 0 \quad (93)$$

$$\int_{-\infty}^{\infty} dx \psi(x)^2 e^{-x^2/2} < \infty \quad (94)$$

Then $\psi(x)$ can be expanded over the basis of Hermite polynomials $H_j^*(x)$, namely

$$\psi(x) = \sum_{j=1}^{\infty} c_j H_j^*(x). \quad (95)$$

Hermite polynomials are defined as $H_1^*(x) = x$, $H_2^*(x) = x^2 - 1$, and the recursive relation $H_{j+1}^*(x) = xH_j^*(x) - jH_{j-1}^*(x)$ for $j \geq 2$. Then $\psi(x)$ is said to have Hermite rank m^* if the first non-vanishing coefficient in the expansion is c_{m^*} , that is $c_j = 0$ for $j < m^*$ and $c_{m^*} \neq 0$.^{117,105}

The following theorem^{102,105,116} then holds for the sum of the variables y_i .

Theorem 9. *Let $\{x_i\}_{i=1, \dots, N}$ be a Gaussian stationary sequence with correlation function decaying at large distance as $r(m) \sim m^{-\alpha}$. Define the sequence $y_i = \psi(x_i)$ with $\psi(x)$ a real function with Hermite rank m^* . If $\alpha < 1/m^*$, the asymptotic distribution of the sum $S_N = \sum_{i=1}^N y_i$ exists, and is non-Gaussian if $m^* > 1$, while it is Gaussian for $m^* = 1$. In the opposite case $\alpha > 1/m^*$, the Gaussian distribution is recovered.*

^cThe definition of a stationary sequence in the general case is actually not obvious, and goes as follows. Let x_1, \dots, x_k denote a set of k random variables. For any vector $\mathbf{i}(k)$ of k integers, $\mathbf{i}(k) = (i_1, \dots, i_k)$, where $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq k$, we note

$$F_{\mathbf{i}(k)}(x_1, \dots, x_k) = \text{Prob}(x_{i_s} < x_s, 1 \leq s \leq k). \quad (90)$$

The sequence $\{x_k\}$ is stationary if the probability $F_{\mathbf{i}(k)}(x_1, \dots, x_k)$ stays invariant if the vector $\mathbf{i}(k)$ is translated by any vector $\mathbf{m} = (m, \dots, m)$, with m any positive integer:

$$F_{\mathbf{i}(k)+\mathbf{m}}(x_1, \dots, x_k) = F_{\mathbf{i}(k)}(x_1, \dots, x_k). \quad (91)$$

Note in particular that i.i.d. random variables form a stationary sequence.

The first part of the theorem is known as Taqqu's reduction theorem.¹¹⁶ The corresponding non-Gaussian limit distributions are known through their cumulants, given by multiple integrals.¹⁰¹ Note that an explicit example with $m^* = 2$ and $\alpha < 1/m^*$ was given by Rosenblatt¹¹⁸ before Theorem 9 was proven. Moreover, this theorem shows that, at least for stationary Gaussian sequences, rather strong correlations can be included without affecting the limit distribution, which remains Gaussian.

From a more general perspective, it seems that when considering generic classes of strongly correlated variables, almost any "reasonable" function could be a particular limit function, so that trying to classify them is hardly possible. For instance, a continuous family of limit functions is obtained in the simple $1/f^\alpha$ -noise model¹⁵¹ of correlated random signals. Still, from a physicist point of view, the knowledge gained from the renormalization group approach in statistical mechanics might suggest that physically relevant classes of strongly correlated random variables may be organized in kinds of universality classes (a notion somehow close to that of basin of attraction appearing in convergence theorems). In this rather optimistic picture, one could a priori guess what kind of probability distribution is related to a particular physical problem, based on general symmetry and dimensionality properties. However, even assuming that such a "gallery" of asymptotic distributions could be defined, it is presently far from being quantitatively completed. Indeed, most of the known results in the physics literature are obtained through perturbative expansions,^{119,120} and the asymptotic distributions are not known exactly in most cases. Moreover, the above gallery of distributions is continuously expanding, through the development of non-equilibrium statistical physics.¹⁵²

4.1.5. *The renormalization group approach*

The renormalization group procedure has had an enormous impact in physics, specifically in the study of critical phenomena, but also in diverse fields of physics like field theory, or the study of disordered systems.¹²¹ The main idea of the renormalization group approach is to coarse-grain step by step the description of the system, while conserving the thermodynamic properties. In more mathematical terms, this means that the original random variables describing the microscopic degrees of freedom are coarse-grained iteratively into "mesoscopic" effective random variables, and that the statistical properties of the sum (for instance the total magnetization in a magnetic system) is preserved. This is due to the fact that the transformation conserves the partition function; since the logarithm of the partition function generates the magnetization cumulants, it follows that this distribution is conserved through the renormalization procedure.

In the following, we shall briefly illustrate on a standard solvable example, the decimation of the Ising chain,¹²² how a renormalization group approach may be used to determine the asymptotic distribution of a sum of correlated variables (in the same spirit, an exact renormalization procedure can be performed in the one-

dimensional XY-model.¹²³) Note that in practice, examples that can be solved exactly through a renormalization group calculation can most often be solved by other more direct methods. Yet, various approximation schemes exist in more complicated situations, and provide valuable insights into the behavior of the system.

Let us also mention that we are not mostly interested here in the critical point of the model, which is a zero-temperature critical point, but rather by the statistics of the magnetization away from the critical point, that is at finite temperature. In this case, the intuitive scaling argument presented above suggests that since the correlation length ξ is finite, the effective number N/ξ of degrees of freedom ($D = 1$ here) diverges in the thermodynamic limit, so that one expects to recover a Gaussian distribution. The decimation procedure allows us to obtain this result explicitly, as through the renormalization process, the joint distribution converges to a factorized distribution (and moments are finite).

The partition function of the Ising chain (or one-dimensional Ising model) with an even number N of sites, and periodic boundary conditions, is given by

$$\mathcal{Z} = \sum_{\{s_i\}} \exp \left(-\beta \sum_{i=1}^N \mathcal{H}_{i,i+1}(s_i, s_{i+1}) \right), \quad (96)$$

where the local Hamiltonian $\mathcal{H}_{i,i+1}(s_i, s_{i+1})$ associated with the link $(i, i + 1)$ is given by

$$\mathcal{H}_{i,i+1}(s_i, s_{i+1}) = -K s_i s_{i+1} - \frac{h}{2}(s_i + s_{i+1}) + c. \quad (97)$$

Note that the role of s_i and s_{i+1} have been symmetrized for later convenience, and that a constant term c has been added. This constant term is irrelevant at this stage and could be set to zero, but such a term will be generated by the renormalization procedure, and it is useful to include it from the beginning.

The basic idea of the decimation procedure is to perform, in the partition function, a partial sum over the spins of –say– odd indices in order to define renormalized coupling constants K' and h' . Then summing over the values of the spins with even indices yields the partition function \mathcal{Z}' of the renormalized model, which is by definition of the renormalization procedure equal to the initial partition function \mathcal{Z} . To be more explicit, one can write \mathcal{Z} as

$$\mathcal{Z} = \sum_{\{s_{2j}\}} \sum_{\{s_{2j+1}\}} \exp \left(-\beta \sum_{i=1}^N \mathcal{H}_{i,i+1}(s_i, s_{i+1}) \right) \quad (98)$$

and rewrite the above equality in the following form:

$$\mathcal{Z} = \sum_{\{s_{2j}\}} \exp \left(-\beta \sum_{j=1}^{N/2} \mathcal{H}'_{j,j+1}(s_{2j}, s_{2j+2}) \right) \quad (99)$$

where $\mathcal{H}'(\{s_{2j}\}) = \sum_{j=1}^{N/2} \mathcal{H}'_{j,j+1}(s_{2j}, s_{2j+2})$ is the renormalized Hamiltonian, defined by

$$\exp[-\beta\mathcal{H}'(\{s_{2j}\})] = \sum_{\{s_{2j+1}\}} \exp\left(-\beta \sum_{i=1}^N \mathcal{H}_{i,i+1}(s_i, s_{i+1})\right). \quad (100)$$

This last relation is satisfied if, for any given $j = 1, \dots, N/2$, and any given values of s_{2j} and s_{2j+2} ,

$$\begin{aligned} & \exp(-\beta\mathcal{H}'_{j,j+1}(s_{2j}, s_{2j+2})) \\ &= \sum_{s_{2j+1}=\pm 1} \exp(-\beta[\mathcal{H}_{2j,2j+1}(s_{2j}, s_{2j+1}) + \mathcal{H}_{2j+1,2j+2}(s_{2j+1}, s_{2j+2})]). \end{aligned} \quad (101)$$

Assuming that $\mathcal{H}'_{j,j+1}(s_{2j}, s_{2j+2})$ takes the form

$$\mathcal{H}'_{j,j+1}(s_{2j}, s_{2j+2}) = -K' s_{2j} s_{2j+2} - \frac{h'}{2}(s_{2j} + s_{2j+2}) + c', \quad (102)$$

one obtains, with the notation $s = s_{2j+1}$,

$$\begin{aligned} & \exp\left(\beta K' s_{2j} s_{2j+2} + \frac{\beta h'}{2}(s_{2j} + s_{2j+2}) - \beta c'\right) \\ &= \sum_{s=\pm 1} \exp\left(\beta K s_{2j} s + \beta K s s_{2j+2} + \frac{\beta h}{2}(s_{2j} + s_{2j+2}) + \beta h s - 2\beta c\right). \end{aligned} \quad (103)$$

Introducing the reduced variables

$$u = e^{-4\beta K}, \quad v = e^{-2\beta h}, \quad \omega = e^{-4\beta c}, \quad (104)$$

Eq. (103) yields the following coupled recursion relations:

$$u' = \frac{u(1+v)^2}{(u+v)(1+uv)}, \quad v' = \frac{v(u+v)}{1+uv}, \quad \omega' = \frac{\omega^2}{u v^2} (1+v)^2 (u+v)(1+uv). \quad (105)$$

One sees that the evolution of u and v , which correspond to the physical coupling constants, is actually decoupled from the evolution of ω which encodes the (apparently useless) constant term in the Hamiltonian.^d When starting from a finite temperature so that $u > 0$ initially, iterations of the renormalization recursion relations leads to one of the fixed point $u_0 = 1$, $v_0 \geq 0$ corresponding to a system with coupling constant $K = 0$ and arbitrary external field h . Since $K = 0$, the spins are independent random variables, so that the CLT applies (the second moment is finite). As the distribution is the same as that of the original system which included a coupling between the spins, one concludes that the magnetization in the correlated system also has a Gaussian distribution in the thermodynamic limit. Note however that one has to assume that the order of the two limits $N \rightarrow \infty$ and infinite number

^dNote however that the evolution of ω under renormalization still contains some useful information, as one can compute from it the free energy at any temperature.

of iterations of the renormalization group can be exchanged, which is not necessarily obvious.

Finally, let us note that the fixed points we have considered here, where the spins become independent random variables, is called a “trivial” fixed point. Generally speaking, what is more interesting from the physicist’s viewpoint is the so-called “critical” fixed point, in which the spins become highly correlated. In the Ising chain however, it corresponds to $u = 0$ and $v = 1$, that is to infinite coupling K , or zero temperature, and zero field h . The absence of finite temperature phase transition is generic in one-dimensional equilibrium systems with short-range interactions.¹²⁴

4.2. Statistics of extreme values

4.2.1. Extreme values of non-identically distributed independent variables

As in the case of statistics of sums of random variables, losing the i.i.d. property leads to much weaker results concerning the limit distributions of extremes. In this section, we focus on sequences of independent random variables $\{x_i\}$, with different marginal cumulative probability distributions F_i . This case has deserved attention recently due to its practical interest, for example in the study of extreme climatic events, where climate changes lead to a modification of the underlying statistics,^{125,126} or in application of ideas of extreme value statistics in evolving risk insurance.⁹⁰ The distribution of the maximum then satisfies

$$F_N^{\max}(z) \equiv \text{Prob}(\max(x_1, \dots, x_N) \leq z) = \prod_{i=1}^N F_i(z), \quad (106)$$

which is an extension of Eq. (28). Note that in practical applications however, the marginal distributions are generally unknown, which makes the above equation essentially useless for practical purposes.

If the random variables are strongly non-identical, the asymptotic distribution of the maximum could be any probability distribution, as shown on a simple example below. Hence, the identification of asymptotic laws for extremes of non-identical random variables has to be done case by case,^{64,60,65,130} and is beyond the scope of the present article.

Some simple understanding of these issues can be gained using an interesting example inspired by Falk and coworkers.⁶⁴ It makes use of the following simple property: if F_0 is a cumulative distribution function, then for any real $\gamma > 0$, the function F_0^γ is also a cumulative distribution function. Hence a simple way to generate non-identical random variables is to choose a cumulative distribution function F_0 and a set of N number $\gamma_i > 0$, and to define a sequence of independent random variables $\{x_i\}$ with cumulative distribution $F_i(z) = F_0(z)^{\gamma_i}$, $i = 1, \dots, N$. Relation (106) then leads to

$$\text{Prob}(\max(x_1, \dots, x_N) \leq z) = F_0(z)^{\sum_{i=1}^N \gamma_i}. \quad (107)$$

It is then clear that the asymptotic law is controlled by the behavior of the sum $S_N = \sum_{i=1}^N \gamma_i$ when $N \rightarrow \infty$. If S_N converges to a finite limit S_∞ , the asymptotic cumulative distribution is simply given by $F_0(z)^{S_\infty}$, which can be any distribution since F_0 is arbitrary.

In contrast, if $S_N \rightarrow \infty$ when $N \rightarrow \infty$, it is rather easy to show that the standard extreme value distributions (namely, the Gumbel, Fréchet and Weibull ones) are obtained. The asymptotic distribution is selected among the three possible ones according to the function F_0 , with the same criteria as those presented in Sect. 3.2.1. For the sake of simplicity, we illustrate this result on the example of the Gumbel distributions, but the same argument holds for Fréchet and Weibull distributions. Defining the function $\zeta(z)$ through $F_0(z) = 1 - \exp(-\zeta(z))$, one then has from Eq. (106)

$$F_N^{\max}(z) = \left(1 - e^{-\zeta(z)}\right)^{S_N}. \quad (108)$$

Since we focus on the Gumbel case, we consistently assume that $\zeta(z) \approx \lambda z^\nu$ when $z \rightarrow \infty$, with $\lambda > 0$ and $\nu > 0$ (although special cases of bounded variables could also be considered within the Gumbel class, as mentioned in Sect. 3.2.1). One would then like to know whether there exists a sequence of real numbers a_N and b_N such that $F_N^{\max}(a_N + b_N x)$ converges to the Gumbel cumulative distribution $H_g(x) = \exp(-e^{-x})$. In the large N limit, we write $\zeta(a_N + b_N x)$ as

$$\zeta(a_N + b_N x) \approx \lambda(a_N + b_N x)^\nu = \lambda a_N^\nu \left(1 + \frac{b_N}{a_N} x\right)^\nu. \quad (109)$$

Let us assume that $b_N/a_N \rightarrow 0$ when $N \rightarrow \infty$, and self-consistently verify this assumption afterwards. Expanding to first order in b_N/a_N , we have

$$\zeta(a_N + b_N x) = \lambda a_N^\nu + \lambda \nu b_N a_N^{\nu-1} x + \mathcal{O}(b_N^2 a_N^{\nu-2}). \quad (110)$$

We choose a_N and b_N such that

$$\lambda a_N^\nu = \ln S_N, \quad b_N a_N^{\nu-1} = \frac{1}{\lambda \nu}. \quad (111)$$

Then, $a_N = (\lambda^{-1} \ln S_N)^{1/\nu}$ diverges with N , and

$$b_N^2 a_N^{\nu-2} = \frac{1}{(\lambda \nu)^2 a_N^\nu} \sim \frac{1}{\ln S_N} \rightarrow 0 \quad (112)$$

when $N \rightarrow \infty$. One then has

$$\zeta(a_N + b_N x) = \ln S_N + x + \mathcal{O}\left(\frac{1}{\ln S_N}\right). \quad (113)$$

One also verifies that the choice of a_N and b_N is consistent with the assumption that $b_N/a_N \rightarrow 0$ when $N \rightarrow \infty$, since $b_N/a_N = 1/(\lambda \nu a_N^\nu)$. Plugging these results into the cumulative distribution F_N^{\max} , one obtains

$$F_N^{\max}(a_N + b_N x) = \left[1 - \frac{1}{S_N} \exp\left(-x + \mathcal{O}\left(\frac{1}{\ln S_N}\right)\right)\right]^{S_N}. \quad (114)$$

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Taking the limit $N \rightarrow \infty$, one finds, as $S_N \rightarrow \infty$,

$$F_N^{\max}(a_N + b_N x) \rightarrow \exp(-e^{-x}) \quad (115)$$

which is nothing but the cumulative Gumbel distribution. Hence one recovers for this specific class of non-identically distributed and independent random variables the standard behavior of i.i.d. variables belonging to the Gumbel class. Note however that the convergence to the asymptotic distribution may be quite slow, due to the logarithmic dependence on S_N .

4.2.2. *Extreme of correlated variables*

The case of extreme value statistics of correlated random variables has deserved attention in recent years, due to its application to physical situations such as the Random Energy Model (REM),^{127,128,129} fluctuating interfaces,^{130,131,132,133,95,134} directed polymers,^{135,136,137} Burgers turbulence,^{85,41} freely expanding gases in one-dimension,¹³⁸ biological evolution of quasispecies¹³⁹ or applied statistics and climatology.^{140,141} A slightly different issue has also been addressed recently, namely the maximum value of a time signal with respect to the initial value.¹⁴² Rigorous mathematical treatments are only available in a very few cases (and more specifically the Gaussian cases^{60,143,64,130}), but approximate treatments, using physicist's tools such as replica trick⁸⁵ or functional renormalization group approaches,¹²⁹ give some indications on the consequence of correlations.

Let us first show using an example⁶⁴ that, as for non-identically distributed variables, any probability distribution could be seen as an asymptotic law for extreme of correlated variables. To that purpose consider a set of i.i.d. random variables $\{y_i\}$ described by an arbitrary distribution, and a random variable v , independent of the variables y_i , described by a cumulative distribution $F(v)$. Now define a new set of random variables x_i through $x_i = v + y_i$ for all i . If there exists a sequence of real numbers $\{a_N\}$ such that

$$\forall \epsilon > 0, \lim_{N \rightarrow \infty} \text{Prob}(|\max(y_1, \dots, y_N) - a_N| > \epsilon) = 0, \quad (116)$$

then in the limit $N \rightarrow \infty$, for any real z :

$$\begin{aligned} \text{Prob}(\max(x_1, \dots, x_N) - a_N \leq z) &= \text{Prob}(v + \max(y_1, \dots, y_N) - a_N \leq z) \\ &\rightarrow \text{Prob}(v \leq z) = F(z). \end{aligned} \quad (117)$$

due to Eq. (116). This is the case for instance if y_i is uniformly distributed over a given interval. We then have an extreme of correlated random variables which is distributed according to an arbitrary probability distribution. This means that as in the case of non-identically distributed random variables, the notion of classes, or of basin of attraction, is less useful than in the i.i.d. case.

Rigorous results can be derived in the particular case of stationary Gaussian sequences of correlated random variables,^{60,143} as defined in Eq. (89), for which $\langle x_i \rangle = 0$ for all i , and the two-point correlation function depends only on $|i - j|$,

namely $\langle x_i x_{i+m} \rangle = r(m)$. Note that the basin of attraction of these theorems is therefore quite small compared to the ones of the CLT or even the generalized CLT. This is a direct consequence of the complexity induced by the loss of the independence between random variables.

The first kind of results deals with the case of constant correlation, such that $\langle x_i x_{i+m} \rangle = r_N$ for all m and i (the value r_N depending on the sample size N only). The extreme distribution depends in this case on the behavior of r_N as a function of N .⁶⁰

Theorem 10. *Let $y_N = \max(x_1, \dots, x_N)$ be the maximum of N elements of a Gaussian sequence $\{x_k\}_{k=1, \dots, N}$ with zero mean, unit variance, and constant correlation $r = r_N$. Let*

$$a_N = \frac{1}{b_N} - \frac{1}{2} b_N (\ln \ln N + \ln 4\pi), \quad b_N = (2 \ln N)^{-1/2}. \quad (118)$$

If, as $N \rightarrow \infty$, $r_N \ln N$ converges to a finite value τ , then the rescaled variable $z = (y_N - a_N)/b_N$ has a limit distribution.

- *If $\tau = 0$, the limit distribution of z is the Gumbel distribution as in the i.i.d. case.*
- *If $\tau > 0$, the limit distribution of z is the convolution of a translated Gumbel distribution $\exp(-(z + \tau) - e^{-(z+\tau)})$, and a Gaussian distribution with zero mean and variance 2τ .*

If $\lim_{N \rightarrow \infty} r_N \ln N = \infty$, then the variable $z' = (y_N - a_N \sqrt{1 - r_N})/\sqrt{r_N}$ has a limit distribution, which is the normal distribution with zero mean and unit variance.

We see that only in the case of weak enough correlation (first case), the asymptotic distribution remains the same as in the i.i.d. case (because the Gaussian distribution of the variables x_i is in the basin of attraction of the Gumbel asymptote). Interestingly, another connection between extreme values and sums seems to appear here, since one finds a Gaussian distribution (typical of sums of i.i.d. random variables) as asymptotic distribution of extreme values of correlated random variables. We shall come back to this issue in Sect. 4.3.

The second kind of results applies to the perhaps more realistic case (at least from a physicist point of view) of a correlation $r(m)$ decreasing with the distance m . The limit distribution of the maximum of the sequence depends on how fast the correlation decays, the transition between the two regimes being once again a logarithmic decay.^{60,129}

Theorem 11. *Let $y_N = \max(x_1, \dots, x_N)$ be the maximum of N elements of a Gaussian sequence $\{x_k\}_{k=1, \dots, N}$ with zero mean, unit variance, and correlation $r(m) = \langle x_k x_{k+m} \rangle$, independent of k . If, as $m \rightarrow \infty$, $r(m) \ln m$ converges to a finite value τ , then the rescaled variable $z = (y_N - a_N)/b_N$ has a limit distribution when $N \rightarrow \infty$, where a_N and b_N are given in Eq. (118).*

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- If $\tau = 0$, the correlation does not affect the limit distribution of the maximum, which is the Gumbel distribution as in the i.i.d. case.
- If $\tau > 0$, the asymptotic distribution of the maximum is the convolution of a translated Gumbel distribution $\exp(-(z + \tau) - e^{-(z+\tau)})$, and a normal distribution with zero mean and variance 2τ .

If $\lim_{m \rightarrow \infty} r(m) \ln m = \infty$ and $\lim_{m \rightarrow \infty} r(m)(\ln m)^{1/3} = 0$, then the variable $z' = (y_N - a_N \sqrt{1 - r(N)}) / \sqrt{r(N)}$ has a limit distribution, which is the normal distribution with zero mean and unit variance.

These results were also derived using functional RG calculations, and applied to the study of the glass transition in the REM with correlated Gaussian potential.¹²⁹

4.3. Relation between statistics of sums and of extreme values

4.3.1. Are there extreme values hidden in $1/f$ -noise?

Let us come back to the $1/f$ -noise problem studied in the previous section, where it was found that the PDF of the integrated power spectrum (or total energy) given by Eqs. (78), (79), is the Gumbel distribution G_1 , while there is a priori no connection with extreme values in this problem. A rather natural idea would be that extreme values may be somehow “hidden” in the computation of certain types of random sums, in the sense that a few terms may dominate the sum. As discussed before, this is the case when considering broad distributions of identically distributed variables – although the distribution of the sum is not in this case an extreme value distribution. In the present situation, the distributions of the individual variables are not broad – the distribution of $u_n = |c_n|^2$ is exponential, as seen in Eq. (80). Still, one might imagine that due to the dispersion of the variances of the different variables $|c_n|^2$, some of the variables (the ones with small n 's) may dominate the sum, and lead to distributions similar to that found in the context of extreme statistics. However, attempts to identify such dominant contributions in problems essentially similar to the $1/f$ -noise, concluded that all terms in the sums are necessary to account for the observed probability distributions. To be more specific, such a study was done in the context of the XY-model, where contributions of the longest wavelength modes were studied,^{144,145} and in the Ising model, where the contribution of the largest connected cluster of –say– up spins was identified.¹⁴⁶ In light of these results, it seems clear that the relation with extreme value statistics, if it exists, is more subtle, and does not come from the dominant contribution of a few terms.

4.3.2. Mapping an extreme value on a sum

Actually, the situation becomes clearer when one takes another perspective. Up to now, we have been looking for extreme values hidden in problems of sums. Let us now take a different point of view: could we instead look for sums hidden in extreme values? Although this question might seem a bit strange at first sight, its

answer actually contains the explanation of the puzzling relation between statistics of extreme values and statistics of some classes of random sums, analogous to the $1/f$ -noise problem.

To see how sums may emerge out of extreme values,^{147,93} let us consider a set of N i.i.d. random variables $x_n > 0$ ($1 \leq n \leq N$), drawn from a distribution $p(x)$. Let us order these variables by relabelling them into y_n in such a way that $y_1 \geq y_2 \geq \dots \geq y_N$. Formally, this means that there exists a permutation σ over the integers $1, \dots, N$ such that $y_n = x_{\sigma(n)}$. By definition, $y_1 = \max(x_1, \dots, x_N)$, and similarly y_k is the k^{th} largest value among the set $\{x_n\}$. Once properly rescaled, the variable y_1 necessarily follows one of the three classes of extreme value statistics given in Theorem 3.

Then, one can introduce the differences between the ordered variables, defining v_n as

$$v_n = y_n - y_{n+1} \quad (1 \leq n \leq N-1), \quad v_N = y_N. \quad (119)$$

It results that the largest value y_1 among the x_n 's can be rewritten as

$$\max(x_1, \dots, x_N) \equiv y_1 = \sum_{n=1}^N v_n. \quad (120)$$

Similarly, the k^{th} largest value y_k can be expressed as

$$y_k = \sum_{n=k}^N v_n. \quad (121)$$

Accordingly, it turns out that extreme values can be quite naturally expressed as sums of random variables. Note that up to now, we have only introduced a formal procedure to recast an extreme value into a sum. To be more quantitative, we need to determine the statistical properties of the variables v_n . To this aim, let us introduce the joint distribution $\Phi_{k,N}(v_k, \dots, v_N)$ of the variables v_k, \dots, v_N . This joint distribution is formally defined as:

$$\begin{aligned} \Phi_{k,N}(v_k, \dots, v_N) &= N! \int_0^\infty dy_N p(y_N) \int_{y_N}^\infty dy_{N-1} p(y_{N-1}) \dots \int_{y_2}^\infty dy_1 p(y_1) \\ &\quad \times \delta(v_N - y_N) \prod_{n=k}^{N-1} \delta(v_n - y_n + y_{n+1}). \end{aligned} \quad (122)$$

From this definition, it is straightforward to show that⁹³

$$\Phi_{k,N}(v_k, \dots, v_N) = \frac{N!}{(k-1)!} \tilde{F} \left(\sum_{i=k}^N v_i \right)^{k-1} \prod_{n=k}^N p \left(\sum_{i=n}^N v_i \right), \quad (123)$$

where the function $\tilde{F}(z)$ is given by

$$\tilde{F}(z) \equiv \int_z^\infty dy p(y). \quad (124)$$

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It is now convenient to perform a shift of indices by introducing $u_n = v_{n+k-1}$, leading to redefine the distribution $\Phi_{k,N}(v_k, \dots, v_N)$ into $\tilde{\Phi}_{k,N'}(u_1, \dots, u_{N'})$, with $N' \equiv N - k + 1$. The expression of $\tilde{\Phi}_{k,N'}(u_1, \dots, u_{N'})$ is immediately deduced from Eq. (123):

$$\tilde{\Phi}_{k,N'}(u_1, \dots, u_{N'}) = \frac{(N' + k - 1)!}{(k - 1)!} \tilde{F} \left(\sum_{i=1}^{N'} u_i \right)^{k-1} \prod_{n=1}^{N'} p \left(\sum_{i=n}^{N'} u_i \right), \quad (125)$$

From the very definition of the variables u_n , it is clear that the sum $\sum_{n=1}^{N'} u_n$, once properly rescaled, converges to one of the asymptotic extreme value distributions g_k , $f_{k,\mu}$ or $w_{k,\mu}$ defined in Eqs. (41), (42) and (43).

It is interesting to note that when describing experimental or numerical data, generalized extreme value distributions, taking k as a real and positive fitting parameter, are often considered.^{40,49} The question then naturally arises to know whether one could give a precise meaning (other than phenomenological fitting functions) to such generalized extreme value distributions.

4.3.3. Generalized extreme value distributions for sums of non-i.i.d. variables

It turns out that the answer to this question is rather straightforward, given the results presented in the previous section. Indeed, from Eq. (125), one sees that the integer k is now a simple parameter of the distribution, and that the distribution $\tilde{\Phi}_{k,N'}(u_1, \dots, u_{N'})$ could easily be generalized to non-integer values of k , provided that factorials are replaced by Gamma functions. In what follows, we shall denote k as a whenever it is not restricted to integer values. Note that one now needs to consider N' as an integer rather than N . In other words, the direct mapping back to the problem of extreme values of i.i.d. random variables is no longer possible for non-integer k (see however the discussion in Sect. 4.3.4). Accordingly, it is necessary to perform an independent calculation⁹³ to show that the sum of the u_n 's converges to the generalized extreme value distributions $g_a(x)$, $f_{a,\mu}(x)$ or $w_{a,\mu}(x)$ defined in Eq. (41) to (43). This calculation has actually been done using a slightly more general distribution $\Psi_{a,N'}(u_1, \dots, u_{N'})$

$$\Psi_{a,N'}(u_1, \dots, u_{N'}) = \frac{1}{Z_{N'}} \Omega \left[\tilde{F} \left(\sum_{n=1}^{N'} u_n \right) \right] \prod_{n=1}^{N'} p \left(\sum_{i=n}^{N'} u_i \right), \quad (126)$$

where $\Omega(t)$ is an arbitrary positive function of $t > 0$, and where the normalization factor $Z_{N'}$ is given by

$$Z_{N'} = \frac{1}{\Gamma(N')} \int_0^1 dt \Omega(t) (1-t)^{N'-1}. \quad (127)$$

In this case, the parameter $a > 0$ characterizing the asymptotic distribution is given by the small t behavior of $\Omega(t)$, namely

$$\Omega(t) \sim t^{a-1}, \quad (t \rightarrow 0). \quad (128)$$

When $\Omega(t) = t^{a-1}$ for $0 < t < 1$, one has

$$Z_{N'} = \frac{\Gamma(a)}{\Gamma(N' + a)}, \quad (129)$$

so that one finds for $\Psi_{a,N'}(u_1, \dots, u_{N'})$ the straightforward generalization of Eq. (123). If $\Omega(t)$ is only asymptotically a power law for $t \rightarrow 0$, one does not recover exactly the same form, but this does not affect the asymptotic distribution which actually depends on the large N' behavior of $Z_{N'}$, itself dominated by the small t behavior of $\Omega(t)$.

To sum up, the distribution of the sum $\sum_{n=1}^{N'} u_n$, with the variables u_n drawn from the joint distribution $\Psi_{a,N'}(u_1, \dots, u_{N'})$, converges to one of the three extreme value distributions, according to the asymptotic large z behavior of the function $p(z)$ appearing in Eq. (126) –and which also defines $\tilde{F}(z)$:

- If $p(z)$ decays at large z faster than any power law, then the asymptotic distribution of the sum is the generalized Gumbel distribution $g_a(x)$.
- If $p(z)$ decays as a power law $p(z) \sim z^{-(1+\mu)}$ ($\mu > 0$) when $z \rightarrow \infty$, the limit distribution is the generalized Fréchet distribution $f_{a,\mu}(x)$.
- If $p(z) = 0$ for all $z > A$ and decays like $p(z) \sim (A - z)^{\mu-1}$ ($\mu > 0$) for $z \rightarrow A^-$, then the limit distribution is the generalized Weibull distribution $w_{a,\mu}(x)$.

These results show that a large class of correlated variables, with non-identical marginal distributions, lead as far as their sum is concerned to the distributions found in extreme value statistics. This fact may explain why these extreme value distributions, or at least distributions qualitatively similar, are often found in correlated systems.

It is also worth mentioning that the distribution $\Psi_{a,N'}(u_1, \dots, u_{N'})$ may be symmetrized by summing over all possible permutations σ of the integers $1, \dots, N'$:

$$\Psi_{a,N'}^{\text{sym}}(u_1, \dots, u_{N'}) = \frac{1}{N'!} \sum_{\text{perm } \sigma} \Psi_{a,N'}(u_{\sigma(1)}, \dots, u_{\sigma(N')}). \quad (130)$$

As the sum $\sum_{n=1}^{N'} u_n$ is invariant under permutation of the terms, the distribution of the sum of random variables described by $\Psi_{a,N'}^{\text{sym}}(u_1, \dots, u_{N'})$ or by $\Psi_{a,N'}(u_1, \dots, u_{N'})$ is the same.

To conclude on this point, let us mention that extreme value distributions with a real index appeared recently in mathematics, in the context of free probability.¹⁴⁸ This topic is far beyond the scope of the present paper, but it could be interesting to see if there is a contact point between the two problems.

To get a more intuitive feeling about the class of random variables defined by the joint probability distribution $\Psi_{a,N'}(u_1, \dots, u_{N'})$, it is interesting to focus on a simple example within the class. Let us consider the case when $p(z)$ is a simple exponential distribution, namely

$$p(z) = \lambda e^{-\lambda z}, \quad z > 0, \quad (131)$$

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leading for $\tilde{F}(z)$ to

$$\tilde{F}(z) = \int_z^\infty dy p(y) = e^{-\lambda z}. \quad (132)$$

Then the distribution $\Psi_{a,N'}(u_1, \dots, u_{N'})$ reads

$$\Psi_{a,N'}(u_1, \dots, u_{N'}) = \prod_{n=1}^{N'} \lambda(n+a-1) e^{-\lambda(n+a-1)u_n}. \quad (133)$$

This is a generalization of the $1/f$ -noise problem. Indeed, defining $u_n = |c_n|^2$ in the $1/f$ -noise model as defined in Eq. (77), one precisely recovers the distribution (133), with $\lambda = \kappa$ and $a = 1$.

4.3.4. Going backward from sums to extreme values

In the previous sections, we have seen how an extreme value problem can be converted into a random sum problem, thanks to the mapping relation (119). A natural question would be to know whether this relation also allows an inverse mapping to be performed, that is to go from a problem of sum to a problem of extreme value, which could lead for instance to Gaussian distributions for extreme value problems, as already observed for instance in problems of fluctuating fronts,¹⁵⁰ and in extreme values of strongly correlated Gaussian sequences (see Theorems 10 and 11). Although to our knowledge this possibility has not been explored yet in the literature, such an inverse mapping can indeed be performed, and we shall briefly sketch the argument in the following.

Consider a set of N positive random variables $\{x_i\}_{i=1,\dots,N}$ with a joint probability distribution $\Psi(x_1, \dots, x_N)$. One can then generate a set of variables $\{z_i\}$ such that

$$z_i = \sum_{j=i}^N x_j \quad (134)$$

The joint distribution $J(z_1, \dots, z_N)$ then reads

$$\begin{aligned} J(z_1, \dots, z_N) &= \int \prod_{i=1}^N dx_i \Psi(x_1, \dots, x_N) \prod_{i=1}^N \delta\left(z_i - \sum_{j=i}^N x_j\right) \\ &= \Psi(z_1 - z_2, z_2 - z_3, \dots, z_{N-1} - z_N, z_N) \end{aligned} \quad (135)$$

However, the variables $\{z_i\}$ generated in this way form an increasing sequence, so that this distribution in some sense “lacks randomness”. This problem can easily be overcome by symmetrizing the distribution, that is by summing over all possible permutations σ over the integers $1, \dots, N$ (only one term in the sum is nonzero):

$$J_{\text{sym}}(z_1, \dots, z_N) = \sum_{\text{perm } \sigma} J(z_{\sigma(1)}, \dots, z_{\sigma(N)}) \prod_{i=1}^{N-1} \Theta(z_{\sigma(i)} - z_{\sigma(i+1)}) \quad (136)$$

so that all variables play a symmetric role. As an illustration, we consider two simple cases: the case when the variables x_i are i.i.d. random variables, and the case when $\Psi(x_1, \dots, x_N)$ is equal to $\Psi_{a,N}(x_1, \dots, x_N)$ defined in Eq. (126). We start with the case of i.i.d. random variables, for which

$$\Psi(x_1, \dots, x_N) = \prod_{i=1}^N p(x_i) \quad (137)$$

If $p(x)$ is such that $\langle x^2 \rangle$ is finite, the maximum value of the set of random variables $\{z_i\}$ drawn from $J_{\text{sym}}(z_1, \dots, z_N)$ is, by construction, asymptotically distributed according to a Gaussian distribution. This is actually reminiscent of Theorems 10 and 11. In the specific case when $p(x)$ is a Gaussian distribution, then $J_{\text{sym}}(z_1, \dots, z_N)$ defines a Gaussian sequence. It would be interesting to know whether this sequence fulfills the hypotheses underlying Theorem 10 or Theorem 11. In contrast, if $\langle x^2 \rangle$ is infinite, the asymptotic distribution of the maximum is a Lévy-stable law.

As for the second case mentioned above, with

$$\Psi(x_1, \dots, x_N) = \Psi_{a,N}(x_1, \dots, x_N), \quad (138)$$

defined in Eq. (126), one finds for $J_{\text{sym}}(z_1, \dots, z_N)$

$$\begin{aligned} J_{\text{sym}}(z_1, \dots, z_N) &= \sum_{\text{perm } \sigma} \frac{1}{Z_N} \Omega \left[\tilde{F}(z_{\sigma(1)}) \right] \left(\prod_{i=1}^N p(z_{\sigma(i)}) \right) \prod_{i=1}^{N-1} \Theta(z_{\sigma(i)} - z_{\sigma(i+1)}) \\ &= \frac{1}{Z_N} \left(\prod_{i=1}^N p(z_i) \right) \sum_{\text{perm } \sigma} \Omega \left[\tilde{F}(z_{\sigma(1)}) \right] \Theta(z_{\sigma(i)} - z_{\sigma(i+1)}) \end{aligned} \quad (139)$$

where the last step is obtained by relabeling $z_{\sigma(i)}$ in the product $\prod_{i=1}^N p(z_{\sigma(i)})$. Then, given the constraints imposed by the Heaviside functions, $z_{\sigma(1)}$ can be replaced by $\max(z_1, \dots, z_N)$, which is independent of the permutation σ . The remaining sum over σ of the product of Heaviside functions is simply equal to one, because for a given set of values $\{z_i\}$, only one permutation σ satisfies all the constraints. As a result, one finds

$$J_{\text{sym}}(z_1, \dots, z_N) = \frac{1}{Z_N} \left(\prod_{i=1}^N p(z_i) \right) \Omega \left[\tilde{F}(\max(z_1, \dots, z_N)) \right] \quad (140)$$

If $\Omega(t)$ is a constant, $J_{\text{sym}}(z_1, \dots, z_N)$ factorizes, and one recovers a set of i.i.d. random variables $\{z_i\}$. This was expected since a constant Ω corresponds to $a = 1$ in the notations of Sect. 4.3.3, that is to standard extreme value statistics of i.i.d. random variables. The more general case $\Omega(t) = \Omega_0 t^{a-1}$ is also interesting; we focus more specifically on the case when $p(z)$ decays faster than any power law when $z \rightarrow \infty$, so that the asymptotic distribution of $\max(z_1, \dots, z_N)$ is the Gumbel $g_a(x)$. In the large a limit, we know that $g_a(x)$ converges to the normal distribution. It is also expected from Eq. (140) that $J_{\text{sym}}(z_1, \dots, z_N)$ depends on $\{z_i\}$ mainly through $\max(z_1, \dots, z_N)$. Going beyond Eq. (140), one might think that if

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$J_{\text{sym}}(z_1, \dots, z_N) = C_N \hat{J}(\max(z_1, \dots, z_N))$, where \hat{J} is an arbitrary function, and C_N is a normalization constant, the asymptotic distribution of the maximum should exactly be Gaussian.^e

4.4. Illustrations on one-dimensional physical models

4.4.1. A stochastic dissipative model

Given that distributions rather similar to the generalized Gumbel distribution have often been reported in the study of rather different physical systems like critical magnetic models,³⁷ turbulent flow experiments,^{37,38} granular gases,⁴⁹ or diverse nonequilibrium models,^{39,40} one could wonder whether some generic physical mechanism is able to generate distributions of the microscopic variables of the type shown in Eq. (126). Indeed, the examples of turbulent flows and granular gases suggest that dissipation may play an important role in the emergence of non-Gaussian statistics of the energy fluctuations. At equilibrium, equipartition of energy implies that degrees of freedom with a quadratic energy all have the same statistical properties, so that the total energy generically has Gaussian fluctuations.

As a purpose of illustration, we consider a simple stochastic model^{147,149} on a one-dimensional finite lattice, in which energy is injected at one boundary, and dissipated in the bulk as well as at the opposite boundary. To be more specific, we label as $n = 1, \dots, N$ the sites of the lattice, and we define on each site n a positive energy variable ε_n . Injection of energy proceeds by adding an amount of energy ν on site $n = 1$ with a probability rate $I(\nu)$. Energy is transferred from site n to site $n + 1$ with a rate $\phi(\nu)$ (note that transport is fully biased), and energy may also be dissipated at site n with a rate $\Delta(\nu)$. Note that the energy transferred from site N is actually dissipated since there is no site $N + 1$.

A case of particular interest, which allows for a simple analytic solution of the model, is when the different transition rates are related in the following way^f

$$I(\nu) = e^{-\beta\nu} \phi(\nu), \quad \Delta(\nu) = (e^{\lambda\nu} - 1) \phi(\nu). \quad (141)$$

The parameters β and λ respectively characterize the intensity of the energy injection and of the energy dissipation with respect to the energy transfer within the system. From the master equation describing the evolution of the system,¹⁴⁷ one can show that the stationary joint distribution $\mathcal{P}_{st}(\{\varepsilon_n\})$ can be expressed as

$$\mathcal{P}_{st}(\{\varepsilon_n\}) = \prod_{n=1}^N (\lambda n + \beta) e^{-(\lambda n + \beta)\varepsilon_n}. \quad (142)$$

Clearly, this is the same distributions as $\Psi_{a,N'}(u_1, \dots, u_{N'})$ given by Eq. (133) provided that one identifies ε_n with u_n , and β with $(a - 1)\lambda$. Hence, it turns out

^eThis property can be verified explicitly in the specific case when $\hat{J}(m) = e^{-\lambda x}$. In this case, the distribution of the maximum is a Gamma distribution of index N , known to converge to a Gaussian distribution when $N \rightarrow \infty$, upon a correct rescaling.

^fNote however that more general cases have also been considered.¹⁴⁹

that the fluctuations of the total energy $E_N = \sum_{n=1}^N \varepsilon_n$ are distributed –again up to a proper rescaling– according to the generalized Gumbel distribution $g_a(x)$ in the limit $N \rightarrow \infty$. The shape parameter a is related to the physical parameters of the model through:

$$a = 1 + \frac{\beta}{\lambda}. \quad (143)$$

This relation emphasizes the role of the dissipation in the statistics of energy fluctuations, since when the dissipation is very small, that is when $\lambda \rightarrow 0$, the shape parameter a of the Gumbel distribution goes to infinity, meaning that the distribution converges to a Gaussian law.

It is interesting to note that another physical interpretation may be given to the parameter a . To see this, we consider the lattice as defined in Fourier space, and we map the sites n of the lattice described above onto wavenumbers $q_n = 2\pi n/L$, where L is the system size (in real space). In the spirit of the $1/f$ -noise problem, one can also interpret the energy ε_n as the squared Fourier amplitude $|c_n|^2$ associated to the wavenumber q_n . One then has

$$\langle |c_n|^2 \rangle = \left(\frac{\lambda q_n L}{2\pi} + \beta \right)^{-1}. \quad (144)$$

On the other hand, the power spectrum $\langle |c_n|^2 \rangle$ is the Fourier transform of the spatial correlation function of the system. Denoting as ξ the correlation length, one can rewrite Eq. (144) in the form

$$\langle |c_n|^2 \rangle = \frac{A}{q_n + \xi^{-1}}, \quad (145)$$

with $A = 2\pi/(\lambda L)$ and

$$\xi = \frac{\lambda L}{2\pi\beta} = \frac{L}{2\pi(a-1)}. \quad (146)$$

This last result is particularly interesting. First, it shows that the correlation length of the system is indeed proportional to the system size L , as expected from the breaking of the central limit theorem. Besides, it turns out that the ratio ξ/L of the correlation length to the system size is directly related to the ratio λ/β comparing dissipation and injection, as well as to the shape parameter a of the Gumbel distribution. Interestingly, in the $1/f$ -noise problem, corresponding to the limit $a \rightarrow 1$, the ratio ξ/L becomes infinite, which may be interpreted as a highly correlated limit. The model described here may then be considered as a model of “truncated $1/f$ -noise”.

4.4.2. A confined gas of classical independent particles

In the last section, we gave an example of physical model leading to the joint distribution (126). However, this example deals with the specific case of independent

(though non-identically distributed) variables. The joint distribution (126) is actually much more general, and it would be interesting to have a simple physical example of correlated random variables following the distribution (126) in its general form.

To this purpose, we consider the simple example of a one-dimensional gas of classical independent particles. Models of one-dimensional gases, like the Jepsen gas,^{153,154,155,156,157} have proved very useful in the development of statistical physics, due to their simplicity and to the possibility to test explicitly some statistical approaches. Here we consider a situation where the gas, subjected to an external potential, is confined within a container closed by a piston (see Fig. 6), and is at thermal equilibrium with a heat reservoir.¹⁵⁸ The positions of the N particles are denoted as $z_1, \dots, z_N > 0$, and the piston is situated at z_p , with the obvious constraint that $z_i < z_p$ for $i = 1, \dots, N$. Particles are subjected to an external potential $U(z)$, such that $U(z) \rightarrow \infty$ when $z \rightarrow \infty$. In addition, the piston is subjected to a potential $U_p(z_p)$, that may be of the same type as that acting on the particles, or of a different nature. For instance, the potential $U_p(z_p)$ may describe a constant force $f_p = -|f_p|$ exerted by an external operator, in which case $U_p(z_p) = |f_p|z_p$, or a harmonic potential $U_p(z_p) = \frac{1}{2}kz_p^2$ created by spring.

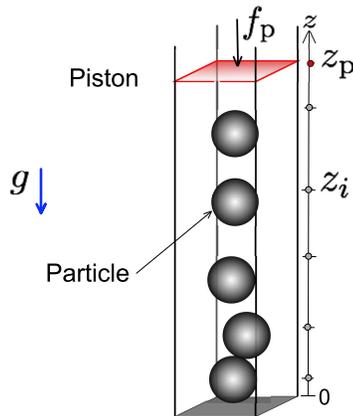


Fig. 6. Simple example: one-dimensional gas of classical independent particles.

At equilibrium at temperature T , the probability distribution of the position of the particles and the piston reads

$$P_N(z_1, \dots, z_N, z_p) = \frac{1}{Z} e^{-\beta U_p(z_p)} \prod_{i=1}^N e^{-\beta U(z_i)} \Theta(z_p - z_i), \quad (147)$$

with $\beta = 1/k_B T$. Similarly to what was done to map extreme values on sum in Eq. (119), one can define the intervals h_i between the ordered positions of the

particles, in such a way that the position of the piston, that is, the volume of the system, can be expressed as

$$z_p = \sum_{j=1}^{N+1} h_j. \quad (148)$$

Eq. (147) is then very similar to the distribution (126), up to a change of N into $N + 1$. The identification is then made by choosing the function $p(z)$ as

$$p(z) = \lambda e^{-\beta U(z)}, \quad (149)$$

(λ is a constant such that $p(z)$ is normalized to one) and by imposing that the function $\Omega(y)$ satisfies

$$\Omega[F(z_p)]p(z_p) = e^{-\beta U_p(z_p)}. \quad (150)$$

A solution to this last equation can be found by noticing that $F(z_p)$ is a monotonous function of z_p , and is thus invertible. We denote as $F^{-1}(y)$ the inverse function of $F(z_p)$. Then the function $\Omega(y)$ can be defined from

$$\Omega(y) = \frac{1}{\lambda} \exp [\beta U(F^{-1}(y)) - \beta U_p(F^{-1}(y))], \quad (151)$$

a relation equivalent to Eq. (150). In the simple case when $U(z) = U_0 z^\alpha$ and $U_p(z_p) = U'_0 z_p^\alpha$, with $U_0, U'_0, \alpha > 0$, the function $\Omega(y)$ behaves as a power law in the limit $y \rightarrow 0$,

$$\Omega(y) \sim y^{a-1}, \quad a = \frac{U'_0}{U_0}, \quad (152)$$

up to logarithmic corrections.¹⁵⁸ Then the distribution of the volume z_p is, in the limit $N \rightarrow \infty$, a generalized Gumbel distribution of parameter a . More general situations can also be considered, leading either to the Gaussian distribution, to the exponential distribution or to the Fréchet or Weibull distributions.¹⁵⁸

4.5. An analog of the Gumbel distribution in two dimensions: the 'BHP' distribution

4.5.1. Extending the 1/f-noise model to higher dimensions

In the previous sections, we saw how, motivated by the study of the 1/f-noise model, extreme values statistics could be reformulated as a problem of sums, allowing interesting extensions to generalized extreme value distributions, as presented in Sect. 4.3.3. In particular, the class of random variables defined by Eq. (126) turns out to be useful in the study of some simple one-dimensional models, as seen in Sect. 4.4. In view of considering more realistic models, it is natural to try to generalize the

$1/f$ -noise model to higher space dimensions.[§] To this aim, one can consider D -dimensional sums of the form

$$E = \sum_{\mathbf{q} \in \mathbb{Z}_D} \mathcal{G}^{-1}(\mathbf{q}) \theta_{\mathbf{q}} \theta_{-\mathbf{q}}, \quad (153)$$

where \mathbb{Z}_D is the D -dimensional Brillouin zone, *i.e.*, the set of D -dimensional vectors $\mathbf{q} = (2\pi n_1/L, \dots, 2\pi n_D/L)$, where $0 \leq n_k \leq L-1$ ($k = 1, \dots, D$), and $\theta_{\mathbf{q}}$ is the Fourier coefficient associated with mode \mathbf{q} . Note that for convenience, we use here notations coming from field theory, but one could equally use the summation on integers as done in Sect. (4.1.2). For simplicity, we search for a model that fulfills rotational invariance (at least at large scale), so that we choose the inverse propagator $\mathcal{G}^{-1}(\mathbf{q})$ to depend only on the modulus $q = |\mathbf{q}|$. An essential property of the $1/f$ -noise model is the logarithmic behavior of the average sum with the number N of terms. One thus wishes to choose the most simple function $\mathcal{G}^{-1}(q)$ that fulfills this property in arbitrary dimension. In the continuous limit, one has

$$\langle E_L \rangle \approx \int_{2\pi/L}^{2\pi} dq \rho(q) \mathcal{G}(q) \quad (154)$$

with $N = L^D$, and where $\rho(q)$ is the coarse-grained density of modes on the reciprocal lattice. One has $\rho(q) \sim q^{D-1}$, so that $E_L \sim \ln L$ on condition that $\mathcal{G}(q) \sim q^{-D}$. We thus simply take $\mathcal{G}^{-1}(q) = q^D$.

The two-dimensional case deserved a lot of attention in the recent literature, since the propagator $\mathcal{G}(q) = q^{-2}$ is very frequent in physical systems. It describes for instance the magnetization of the two-dimensional XY-model¹⁵⁹ in the low-temperature regime, as well as the roughness of fluctuating interfaces in the two-dimensional Edwards-Wilkinson model.^{160,20,40} The corresponding distribution is often called "BHP distribution" after Bramwell, Holdsworth and Pinton who first reported it in turbulent flows and in the XY-model.³⁷

In the context of the two-dimensional XY-model, the global quantity of interest is the total magnetization M of the sample of N spins, defined in terms of independent Fourier coefficients $\theta_{\mathbf{q}}$ by

$$M = 1 - \frac{1}{N} \sum_{\mathbf{q} \in \mathbb{Z}_D} \theta_{\mathbf{q}} \theta_{-\mathbf{q}}. \quad (155)$$

Bramwell and co-workers obtained an expression for the Fourier transform of the BHP distribution,⁴⁰ under some hypotheses that have been questioned and clarified later on.^{164,165} Although the BHP distribution is not of Gumbel type, it has been shown that it could be quite well approximated (for instance by matching the first four cumulants) by a generalized Gumbel distribution with parameter $a \approx 1.57$, in

[§]Note that interesting one-dimensional generalizations, known as $1/f^\alpha$ -noise, have also been studied.^{151,152}

the experimentally or numerically accessible window^{39,40} (see Fig. 7). This similarity with the (generalized) Gumbel distribution is confirmed by the asymptotic expressions of the tails:⁴⁰

$$P_{\text{BHP}}(x) \sim x e^{\lambda_1 x}, \quad x \rightarrow -\infty \quad (156)$$

$$P_{\text{BHP}}(x) \sim \exp(-\eta e^{\lambda_2 x} + \lambda_2 x) \quad x \rightarrow +\infty, \quad (157)$$

with some positive constants η , λ_1 and λ_2 . Note that because of the minus sign in the definition of the magnetization (155), the asymmetry is opposed to the one of the standard extreme value distributions presented in section (3.2.1). These asymptotic relations are interesting, since they show at the same time that the tails are qualitatively similar to that of the Gumbel distribution, but that there exists quantitative difference, like the algebraic prefactor in the positive tail.

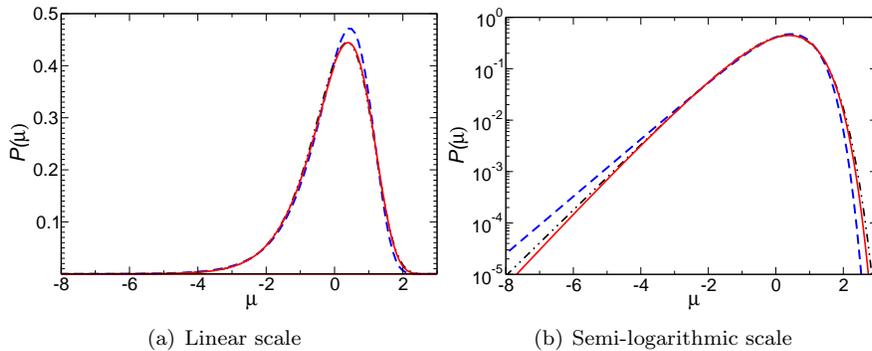


Fig. 7. Comparison between standard Gumbel distribution ($a = 1$, dash), generalized Gumbel distribution with $a = 1.57$ (dot dash), and BHP distribution (plain).

4.5.2. BHP distribution as the leading contribution in the moderately correlated regime

It has been early noticed that the BHP distribution also appears, within experimental errors, in a context completely different from magnetic systems, namely the fluctuations of injected power in a three-dimensional driven turbulent flow.³⁷ This amazing observation triggered many other studies where the BHP distribution has been found to be a good approximate description of experimentally or numerically measured distributions.^{38,41,43,45,48,49} The possible reasons for this apparent ubiquity have been quite debated in the last decade,^{161,162,163,147,93} particularly due to a tentative link with extreme events.

We now give a generic argument, inspired by the study of the two-dimensional Ising model,^{146,166} in order to clarify this issue. Consider a system fluctuating around a local minimum of potential energy; this could be a system at equilibrium,

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or a particular case of out-of-equilibrium system. The distribution of the fluctuations of the global quantity is described by a (pseudo-)Hamiltonian, or 'action' in a field-theoretic language, which could be quite complex. In order to obtain a Gaussian effective action, the simplest idea is to perform a perturbative expansion. From the probability viewpoint, this is similar to group the random variables by blocks of linear size of the order of the correlation length. It is however well-known from the theory of critical phenomena that the resulting effective action does not describe correctly all the system properties. The breakdown of this expansion is quantified by the Ginzburg criterion.¹⁶⁷ We define the ratio

$$R_\xi = \frac{\int_{S(\xi)} C(\mathbf{r}) d\mathbf{r}}{\int_{S(\xi)} \langle m(\mathbf{r}) \rangle^2 d\mathbf{r}}, \quad (158)$$

where ξ is the correlation length, $C(\mathbf{r})$ the two-point correlation function and $m(\mathbf{r})$ the local order parameter (to be interpreted as the local magnetization in the Ising model); the integrals are over a sphere of radius ξ in D dimensions. A mean-field theory captures the physical behavior of the system if $R_\xi \ll 1$.¹⁶⁷ In the present context, we do not look for an effective description of all the properties of the system at the scale of the correlation length, but we only want to describe the behavior of its large scale fluctuations. A natural extension of the previous criterion in order to check the validity of a perturbative approach at the integral scale is then¹⁶⁶

$$R_L = \frac{\int_{S(L)} C(\mathbf{r}) d\mathbf{r}}{\int_{S(L)} \langle m(\mathbf{r}) \rangle^2 d\mathbf{r}} = A \left(\frac{\xi}{L} \right)^D \ll 1, \quad (159)$$

where A is an order unity constant. This criterion is actually much less restrictive than the original criterion. This means that the large scale behavior of a system could be studied by a perturbative expansion much more easily than the small scale behavior. In particular, for the two-dimensional Ising model, it is possible to perform such an expansion while the (finite size) system is already in the critical region.¹⁶⁶ starting in the low temperature regime, we approach the critical point from below, so that ξ increases. We then increase the system size while keeping the ratio ξ/L constant, but much less than 1. Doing so the correlation length diverges, confirming the critical behavior of the system considered. But it diverges with a small amplitude which allows the large scale behavior to be captured by a perturbation theory. The action can then be decomposed in the following way:

$$\mathcal{S} = \frac{1}{2} \sum_{\mathbf{q} \neq 0} \mathcal{G}^{-1}(\mathbf{q}, \phi_0) \theta_{\mathbf{q}} \theta_{-\mathbf{q}} + \mathcal{S}_0(\phi_0), \quad (160)$$

where

$$\mathcal{G}^{-1}(\mathbf{q}, \phi_0) = \phi_0 + q_x^2 + q_y^2, \quad (161)$$

and ϕ_0 is the position of the energy minimum. A detailed study of those fluctuations,^{166,168} shows that the first part of the action leads to fluctuations similar to that of the two-dimensional XY-model at low temperature, with a cut-off

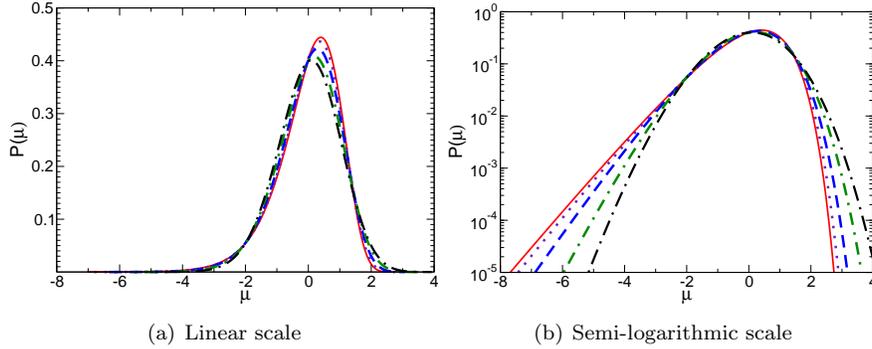


Fig. 8. Distributions obtained for the Gaussian model using propagator (161), for different masses: $\phi_0 = 0$ ('BHP' distribution, plain), $\phi_0 = 0.1$ (dot), $\phi_0 = 1$ (dash), $\phi_0 = 2$ (dot dash) and $\phi_0 = 5$ (long dash). Note the convergence towards the normal distribution when the masse increases.

related to ϕ_0 , playing a role similar to that of a magnetic field in the XY-model.¹⁶⁹ This contribution does not depend on the universality class (in the sense of critical phenomena) of the system, and therefore could be seen as "super-universal"; it depends only on the space dimension D . If ϕ_0 can be varied by adjusting the external control parameters (like temperature or magnetic field), reducing its value close to zero should make the distribution closer to the BHP one, as illustrated on Fig. 8. However, doing so, the effect of higher order terms (in particular quartic terms) in the action may become significant, leading to corrections with respect to the BHP distribution. These corrections correspond to the fact that the system could leave the local minima and explore the whole phase space; they are characteristic of the system considered, and can be formally computed using instantons.¹⁷⁰

The contribution of these corrections in the probability distribution is actually rather easy to identify. By construction the large negative fluctuations induced by $\mathcal{G}^{-1}(\mathbf{q}, \phi_0)$ are closer to the Gaussian distribution than to the BHP distribution, given by $\mathcal{G}^{-1}(\mathbf{q}, 0)$. In particular the quasi-exponential tail should be below the one of the BHP distribution, as seen on Fig. 9, which compares numerical simulations of the two-dimensional Ising model and of the two-dimensional Gaussian model. At low temperature, Fig. 9(a), the two distributions overlap, showing that non-linear contributions are negligible. While we approach the critical point from below, Fig. 9(b), the weight of such corrections increases, so that it becomes impossible to neglect them to describe the large fluctuations. These observations lead to a practical criterion to say whether the large scale behavior of the system could be studied through a perturbative approach.

Altogether, the above approach –which can be generalized in a straightforward way to D -dimensional systems– yields a rationale for the frequent appearance of distributions close to the BHP distribution in two-dimensional systems. This comes from the rather general relevance of Gaussian models with propagator $\mathcal{G}^{-1}(\mathbf{q}, 0)$ in

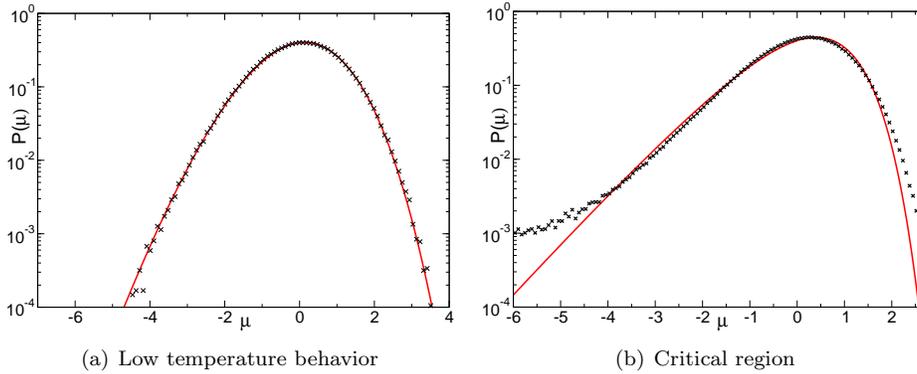
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Fig. 9. (a) Ising model at $T = 1.7J$ for $N = 64^2$ (cross), and Gaussian model with mass $\phi_0 = 0.3$ (plain). (b) Ising model at $T = 2.13J$ for $N = 64^2$ (cross), and BHP distribution (plain), corresponding to a 2D gaussian model with zero mass.

the description of fluctuations around a local minimum of energy. Note however that, quite importantly, this does not explain the observation of the BHP distribution in turbulent flows: assuming that a Gaussian action could be a reasonable description of such a flow (which would remain to be justified), one would expect a three-dimensional Gaussian model to be relevant in this case. This might suggest that a dimensional reduction is at play in out-of-equilibrium systems. Alternatively, it may be possible that a Gaussian action is not a relevant description in this case, and that the reason for the appearance of an approximate BHP distribution in this context relies on completely different grounds. In any case, understanding the statistical behavior of driven systems, and the possible relationships with (critical) equilibrium systems, remains one of the challenges of nonequilibrium statistical physics.

5. Conclusion

In the present review article, we tried to adopt a somehow original perspective by presenting essentially on the same footing different types of convergence theorems, namely those for sums of random variables and those for extremes values. The motivations for this was to highlight the similarities, differences and relationships between these two a priori unrelated fields of probability and statistics. As this review paper is mainly aimed at a physicist readership, we attempted to incorporate both the mathematical rigor when stating the theorems, and more intuitive physicist's approaches when going farther than precise mathematical results would allow us to go. We also tried to give many physical examples of applications, in order to make contact between formal results and the real world, or at least some simplified pictures of the latter.

Beside recalling the standard mathematical convergence theorems, one of the

central points of the present paper is the mapping between extreme values and sums of random variables, which precisely allows one to bridge the gap between the two fields. Although this mapping is very simple, it has non trivial consequences, in that it allows to map the convergence theorems known in one field to the other field. In particular, generalized extreme value distributions can be recovered for sums of specific classes of non-i.i.d. variables, and Gaussian distributions are found for the extreme value of some classes of correlated random variables. Another bridge between the two fields is also the dominant effect of the largest terms within sums of broadly distributed variables. Still, in this case, the relation is very simple (and more direct), and relies on the typical magnitude of the extreme values rather than on the specific properties of their fluctuations. Moreover, the sum does not have exactly the same distribution as its largest term, only the scaling properties with the number of terms is the same.

Clearly, statistical physics remains a fertile testing ground for probability concepts, and both fields benefit from this mutual interaction. In this spirit, let us mention a recent example motivated by the physics of disordered systems, and more specifically the Random Energy Model.^{127,128} In this model, the partition function, which is a random variable due to the presence of random disorder, is a sum of independent random contributions (the Boltzmann-Gibbs weights associated to microscopic configurations). One of the interests of this problem comes from the fact that the statistics of each term depends on the total number of terms, both being related to the system size. Hence standard convergence theorems cannot be applied, and this induces a non-trivial behavior, which is at the origin of the glass transition in this model. Recent progresses have been made in this specific field,¹⁷¹ and one may hope that such results will trigger new researches both in mathematics and physics.

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References

1. E. W. Montroll and H. Scher, *J. Stat. Phys.* **9**, 101 (1973).
2. M. F. Shlesinger, *J. Stat. Phys.* **10**, 421 (1974).
3. H. Scher and E. W. Montroll, *Phys. Rev.* **B12**, 2455 (1975).
4. M. F. Shlesinger, *Ann. Rev. Phys. Chem.* **39**, 269 (1988).
5. J.-P. Bouchaud and A. Georges, *Phys. Rep.* **195**, 127 (1990).
6. J.-P. Bouchaud, *J. Phys. I (France)* **2**, 1705 (1992).
7. J.-P. Bouchaud and D. S. Dean, *J. Phys. I (France)* **5**, 265 (1995).
8. F. Bardou, J.-P. Bouchaud, O. Emile, A. Aspect, and C. Cohen-Tannoudji, *Phys. Rev. Lett.* **72**, 203 (1994).
9. F. Bardou, J.-P. Bouchaud, A. Aspect, and C. Cohen-Tannoudji, *Lévy Statistics and Laser Cooling* (Cambridge University Press, Cambridge, 2002).

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10. M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, *Nature* **363**, 31 (1993).
11. T. H. Solomon, E. R. Weeks, and H. L. Swinney, *Phys. Rev. Lett.* **71**, 3975 (1993).
12. I. A. Min, I. Mezic, A. Leonard, *Phys. Fluids* **8**, 1169 (1996).
13. X. Brokmann, J.-P. Hermier, G. Messin, P. Desbiolles, J.-P. Bouchaud, and M. Dahan, *Phys. Rev. Lett.* **90**, 120601 (2003).
14. A. D. Bruce, *J. Phys.* **C14**, 3667 (1981).
15. K. Binder, *Z. Phys.* **B43**, 119 (1981).
16. K. Binder, *Computational Methods in Field Theory* (Springer, Berlin, 1992).
17. B. Zheng and S. Trimper, *Phys. Rev. Lett.* **87**, 188901 (2001).
18. B. Zheng, *Phys. Rev.* **E67**, 026114 (2003).
19. M. Plischke, Z. Rácz, and R. K. P. Zia, *Phys. Rev.* **E50**, 3589 (1994).
20. Z. Rácz and M. Plischke, *Phys. Rev.* **E50**, 3530 (1994).
21. M. K. Verma, *Physica* **A277**, 359 (2000).
22. E. Marinari, A. Pagnani, G. Parisi, and Z. Rácz, *Phys. Rev.* **E65**, 026136 (2002).
23. A. Rosso, W. Krauth, P. Le Doussal, J. Vannimenus, and K. J. Wiese, *Phys. Rev.* **E68**, 036128 (2003).
24. S. Moulinet, A. Rosso, W. Krauth, and E. Rolley, *Phys. Rev.* **E69**, 035103 (2004).
25. U. Frisch, *Turbulence* (Cambridge Univ. Press, Cambridge, 1995).
26. R. Friedrich and J. Peinke, *Phys. Rev. Lett.* **78**, 863 (1997).
27. B. A. Carreras *et al.*, *Phys. Rev. Lett.* **83**, 3653 (1999).
28. B. Portelli, P. C. W. Holdsworth, and J.-F. Pinton, *Phys. Rev. Lett.* **90**, 104501 (2003).
29. L. Chevillard, S. G. Roux, E. Levêque, N. Mordant, J.-F. Pinton, and A. Arnéodo, *Phys. Rev. Lett.* **95**, 064501 (2005).
30. S. Aumaitre, S. Fauve, S. McNamara, and P. Poggi, *Eur. Phys. J.* **B19**, 449 (2001).
31. J. Farago, *J. Stat. Phys.* **107**, 781 (2002).
32. F. Coppex, M. Droz, J. Piasecki, and E. Trizac, *Physica* **A329**, 114 (2003).
33. P. Visco, A. Puglisi, A. Barrat, F. van Wijland, and E. Trizac, *Eur. Phys. J.* **B51**, 377 (2006).
34. R. Weill, A. Rosen, A. Gordon, O. Gat, and B. Fischer, *Phys. Rev. Lett.* **95**, 013903 (2005).
35. S. Pilgram, A. N. Jordan, E. V. Sukhorukov, and M. Büttiker, *Phys. Rev. Lett.* **90**, 206801 (2003).
36. B. Reulet, J. Senzier, and D. E. Prober, *Phys. Rev. Lett.* **91**, 196601 (2003).
37. S. T. Bramwell, P. C. W. Holdsworth and J.-F. Pinton, *Nature* **396**, 552 (1998).
38. J.-F. Pinton, P. C. W. Holdsworth and R. Labbé, *Phys. Rev.* **E60**, R2452 (1999).
39. S. T. Bramwell *et al.* *Phys. Rev. Lett.* **87**, 188902 (2001).
40. S. T. Bramwell *et al.*, *Phys. Rev.* **E63** 041106 (2001).
41. A. Noullez and J.-F. Pinton, *Eur. Phys. J.* **B28**, 231 (2002).
42. S. T. Bramwell, T. Fennell, P. C. W. Holdsworth, and B. Portelli, *Europhys. Lett.* **57**, 310 (2002).
43. T. Tóth-Katona and J. Gleeson, *Phys. Rev. Lett.* **91**, 264501 (2003).
44. A. A. Fedorenko and S. Stepanow, *Phys. Rev.* **E68**, 056115 (2003).
45. C. Pannetta *et al.*, *Semicond. Sci. Technol.*, **19**, S164 (2004).
46. C. Chamon *et al.*, *J. Chem. Phys.* **121**, 10120 (2004).
47. F. van Wijland, *Physica* **A332**, 360 (2004).
48. B. Ph. van Milligen *et al.*, *Phys. of Plasmas* **12**, 052507 (2005).
49. J. J. Brey, M. I. Garcia de Soria, P. Maynar and M. J. Ruiz-Montero, *Phys. Rev. Lett.* **94**, 098001 (2005).
50. A. Duri, H. Bissig, V. Trappe and L. Cipelletti, *Phys. Rev. E* **72**, 051401 (2005).
51. P. A. Varotsos, N. V. Sarlis, H. K. Tanaka and E. S. Skordas, *Phys. Rev.* **E72**, 041103

- (2005).
52. R. P. Feynman, *The Character of Physical Laws*, (MIT Press, 1967).
 53. W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966), Vol. I.
 54. B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, 1954).
 55. P. Lévy, *Théorie de l'Addition de Variables Aléatoires* (Gauthier-Villard, Paris, 1954; J. Gabay, 2003).
 56. W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966), Vol. II.
 57. V. V. Petrov, *Sums of Independent Random Variables* (Springer-Verlag, Berlin, 1975).
 58. V. V. Petrov, *Limit Theorems of Probability Theory, Sequences of Independent Random Variables* (Oxford University Press, Oxford, 1995).
 59. B. Gnedenko, *Ann. Math.* **44**, 423 (1943).
 60. J. Galambos, *The Asymptotic Theory of Extreme Order Statistics* (Wiley, New York, 1987).
 61. E. J. Gumbel, *Statistics of Extremes* (Columbia University Press, New York, 1958; Dover Publication, 2004).
 62. P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events* (Springer, Berlin, 1997).
 63. R. Reiss and M. Thomas, *Statistical Analysis of Extreme Values* (Birkhäuser, Basel, 2001).
 64. M. Falk, J. Hüsler and R.-D Reiss, *Laws of Small Numbers: Extremes and Rare Events* (Birkhäuser, Basel, 2004).
 65. S. Coles, *An Introduction to Statistical Modeling of Extreme Values*, Springer Series in Statistics (Springer-Verlag London, Berlin, Heidelberg 2001).
 66. G. Gyorgyi, N. R. Moloney, K. Ozogany, Z. Rácz, arXiv:0712.3993.
 67. G. Jona-Lasinio, *Physics Reports* **352**, 439-458 (2001).
 68. H. Poincaré, *Thermodynamique* (Georges Carré, Paris, 1892), cited by J. H. Gaddum, *Nature* **156**, 463 (1945).
 69. G. Zumofen and J. Klafter, *Chem. Phys. Lett.* **219**, 303 (1994).
 70. E. Barkai, A. V. Naumov, Y. G. Vainer, M. Bauer, and I. Kador, *Phys. Rev. Lett.* **91**, 075502 (2003).
 71. A. Ott, J.-P. Bouchaud, D. Langevin, and W. Urbach, *Phys. Rev. Lett.* **65**, 2201 (1990).
 72. S. Marksteiner, K. Ellinger, and P. Zoller, *Phys. Rev.* **A53**, 3409 (1996).
 73. H. Katori, S. Schlipf, and H. Walther, *Phys. Rev. Lett.* **79**, 2221 (1997).
 74. C. Monthus and J.-P. Bouchaud, *J. Phys.* **A29**, 3847 (1996).
 75. E. Bertin and J.-P. Bouchaud, *J. Phys.* **A35**, 3039 (2002).
 76. B. Rinn, P. Maass, and J.-P. Bouchaud, *Phys. Rev. Lett.* **84**, 5403 (2000).
 77. E. M. Bertin and J.-P. Bouchaud, *Phys. Rev.* **E67**, 026128 (2003).
 78. E. M. Bertin and J.-P. Bouchaud, *Phys. Rev.* **E67**, 065105(R) (2003).
 79. C. Monthus, *Phys. Rev.* **E68**, 036114 (2003).
 80. E. Bertin and F. Bardou, *Am. J. Phys.* **76**, 630 (2008).
 81. S. Blinnikov and R. Moessner, *Astron. Astrophys. Suppl. Ser.* **130**, 193 (1998).
 82. V. da Costa, Y. Henry, F. Bardou, M. Romeo, and K. Ounadjela, *Eur. Phys. J.* **B13**, 297 (2000).
 83. F. Bardou, *Europhys. Lett.* **39**, 239 (1997).
 84. M. Romeo, V. da Costa and F. Bardou, *Eur. Phys. J.* **B32**, 513 (2003).
 85. J.-P. Bouchaud and M. Mézard, *J. Phys.* **A30**, 7997 (1997).

56 *Maxime Clusel, Eric Bertin*

86. A. Baldassarri, A. Gabrielli and B. Sapoval, *Europhys. Lett.* **59**, 232 (2002).
87. R. W. Katz, M. B. Parlangi and P. Naveau, *Advances in Water Resources* **25**, 1287 (2002).
88. D. Sornette, L. Knopoff, Y. Kagan and C. Vannest, *J. Geophys. Res.* **101**, 13883 (1996).
89. F. Longin, *J. Bank. Finance* **24**, 1097 (2000).
90. J.-P. Bouchaud and M. Potters, *Theory of Financial Risk and Derivative Pricing* (Cambridge University Press, Cambridge, 2003), 2nd edition.
91. R. von Mises, reprinted in *Selected Papers II*, Amer. Math. Soc., Providence, R.I. (1954).
92. T. Antal, M. Droz, G. Györgyi and Z. Rácz, *Phys. Rev. Lett.* **87**, 240601 (2001).
93. E. Bertin and M. Clusel, *J. Phys.* **A39**, 7607 (2006).
94. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, London, 5th edition, 1994).
95. G. Schehr and S. N. Majumdar, *Phys. Rev.* **E73**, 056103 (2006).
96. A. Comtet, P. Leboeuf, and S. N. Majumdar, *Phys. Rev. Lett.* **98**, 070404 (2007)
97. P. Billingsley, *Proc. Am. Math. Soc.* **12**, 788 (1961).
98. I. A. Ibragimov, *Theo. Proba. Appl.* **8**, 83 (1963).
99. T. C. Sun, *J. Math. and Mech.* **14**, 71 (1965).
100. M. S. Taqqu, *Z. Wahrsch. Verw. Gebiete* **40**, 203 (1977).
101. R. L. Dobrushin and P. Major, *Z. Wahrsch. Verw. Gebiete* **50**, 27 (1979).
102. M. S. Taqqu, *Z. Wahrsch. Verw. Gebiete* **50**, 53 (1979).
103. M. Rosenblatt, *Z. Wahrsch. Verw. Gebiete* **55**, 123 (1981).
104. P. Major, *Z. Wahrsch. Verw. Gebiete* **57**, 129 (1981).
105. P. Breuer and P. Major, *J. Multiv. Anal.* **13**, 425 (1983).
106. F. Baldovin and A. L. Stella, *Phys. Rev.* **E75**, 020101(R) (2007).
107. S. Umarov, C. Tsallis, and S. Steinberg, cond-mat/0603593.
108. S. Umarov, C. Tsallis, M. Gell-Mann, and S. Steinberg, cond-mat/0606038.
109. S. Umarov, C. Tsallis, M. Gell-Mann, and S. Steinberg, cond-mat/0606040.
110. A. Pluchino, A. Rapisarda, and C. Tsallis, *E.P.L.* **80**, 26002 (2007).
111. V. Latora, A. Rapisarda, and C. Tsallis, *Phys. Rev.* **E64**, 056134 (2001).
112. A. Rapisarda and A. Pluchino, *Europhys. News* **36**, 202 (2005).
113. F. Bouchet, T. Dauxois, and S. Ruffo, *Europhys. News* **37**, 9 (2006).
114. H. J. Hilhorst and G. Schehr, *J. Stat. Mech.* P06003 (2007).
115. R. S. Ellis, *Entropy, Large Deviations and Statistical Mechanics*, (Springer, New York, 1985).
116. R. Botet and M. Płoszajczak, *Universal Fluctuations –The Phenomenology of Hadronic Matter* (World Scientific, Singapore, 2002).
117. M. S. Taqqu, *Z. Wahrsch. Verw. Gebiete* **31**, 287 (1975).
118. M. Rosenblatt, *Proceedings of the 4th Berkeley Symposium on Math. Stat. and Prob.* (University of California, 1961).
119. K. Binder, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, London, 1987), Vol. 8, p. 1, and references therein.
120. H. W. Diehl, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, London, 1987), Vol. 10, p. 76, and references therein.
121. K.J. Wiese, *Ann. Henri Poincaré* **4**, 473 (2003).
122. D. R. Nelson and M. E. Fisher, *Ann. Phys* **91**, 226 (1975).
123. M. Droz and A. Malaspinas, *J. Phys.* **C11**, 2729 (1978).
124. E. H. Lieb and D. C. Mattis, *Mathematical Physics in One Dimension: Exactly Soluble Models of Interacting Particles* (Academic Press, London, 1966).

125. P. Naveau, M. Nogaj, C. Ammann, P. Yiou, D. Cooley, and V. Jomelli, *C. R. Acad. Sci.* **337**, 1013 (2005).
126. D. Cooley, P. Naveau, and V. Jomelli, *Environmetrics* **17**, 555 (2006)
127. B. Derrida, *Phys. Rev. Lett.* **45**, 79 (1980).
128. B. Derrida, *Phys. Rev.* **B24**, 2613 (1981).
129. D. Carpentier et P. Le Doussal, *Phys. Rev.* **E63**, 026110 (2001).
130. G. Györgyi, P. C. W. Holdsworth, B. Portelli, and Z. Rácz, *Phys. Rev.* **E68**, 056116 (2003).
131. S. N. Majumdar and A. Comtet, *Phys. Rev. Lett.* **92**, 225501 (2004).
132. S. N. Majumdar and A. Comtet, *J. Stat. Phys.* bf 119, 777 (2005).
133. D. S. Lee, *Phys. Rev. Lett.* **95**, 150601 (2005).
134. G. Györgyi, N. R. Moloney, K. Ozogány, and Z. Rácz, *Phys. Rev.* **E75**, 021123 (2007).
135. K. Johansson, *Commun. Math. Phys.* **209**, 437 (2000).
136. S. N. Majumdar and P. L. Krapivsky, *Phys. Rev.* **E62**, 7735 (2000).
137. D. S. Dean and S. N. Majumdar, *Phys. Rev.* **E64**, 046121 (2001).
138. I. Bena and S. N. Majumdar, *Phys. Rev.* **E75**, 051103 (2007).
139. J. Krug and C. Carl, *Physica* **A318**, 137 (2003).
140. M. Schlather and J. Tawn, *Biometrika*. **90**, 139 (2003).
141. D. Cooley, P. Naveau, and P. Poncet, “Variograms for spatial max-stable random fields”, in *Dependence in Probability and Statistics* (Springer, Berlin, 2006).
142. T. W. Burkhardt, G. Györgyi, N. R. Moloney, and Z. Rácz, *Phys. Rev.* **E76**, 041119 (2007).
143. A. Baldassarri, *Statistics of Persistent Extreme Events*, Ph.D. Thesis, Univ. Paris XI, Paris, 1999.
144. B. Portelli and P. C. W. Holdsworth, *J. Phys.* **A35**, 1231 (2002).
145. B. Portelli, *Fluctuations of Global Quantities in Correlated Systems*, Ph.D. Thesis, École normale supérieure de Lyon, Lyon, 2001.
146. M. Clusel, J.-Y. Fortin and P. C. W. Holdsworth, *Phys. Rev.* **E70**, 046112 (2004).
147. E. Bertin, *Phys. Rev. Lett.* **95**, 170601 (2005).
148. G. Ben Arous and D. V. Voiculescu, *Ann. Probab.* **34**, 1643 (2006).
149. E. Bertin, *J. Phys.* **A39**, 1539 (2006).
150. D. Panja, *Phys. Rev.* **E70**, 036101 (2004).
151. T. Antal, M. Droz, G. Györgyi and Z. Rácz, *Phys. Rev.* **E65**, 046140 (2002).
152. Z. Rácz, in *Proceedings of SPIE 5112*, ed. M. B. Weissman, N. E. Israeloff, A. Shulim Kogan, (2003), p. 248.
153. H. L. Frisch, *Phys. Rev.* **104**, 1 (1956).
154. D. W. Jepsen, *J. Math. Phys.* **6**, 405 (1965).
155. J. L. Lebowitz and J. K. Percus, *Phys. Rev.* **155**, 122 (1967).
156. J. Piasecki, *J. Stat. Phys.* **104**, 1145 (2001).
157. V. Balakrishnan, I. Bena, and C. Van den Broeck, *Phys. Rev. E* **65**, 031102 (2002).
158. E. Bertin, M. Clusel, and P. C. W. Holdsworth, preprint arXiv:0803.4149, to appear in *J. Stat. Mech.* (2008).
159. P. Archambault, S. T. Bramwell and P. C. W. Holdsworth, *J. Phys.* **A30**, 8363 (1997).
160. S. F. Edwards and D. R. Wilkinson, *Proc. Roc. Soc. London* **A381**, 17 (1982).
161. K. Dahlstedt and H. J. Jensen, *J. Phys.* **A34**, 11193 (2001).
162. N. W. Watkins, S. C. Chapman and G. Rowlands, *Phys. Rev. Lett.* **89**, 208901 (2002).
163. S. T. Bramwell *et al.*, *Phys. Rev. Lett.* **89**, 208902 (2002).
164. S. T. Banks and S. T. Bramwell, *J. Phys.* **A38**, 5603 (2005).
165. G. Mack, G. Palma and L. Vergara, *Phys. Rev.* **E72**, 026119 (2005).

58 *Maxime Clusel, Eric Bertin*

166. M. Clusel, J.-Y. Fortin and P. C. W. Holdsworth, *Europhys. Lett.* **76**, 1008 (2006).
167. N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group Approach* (Perseus Books, Reading, 1992).
168. M. Clusel, *Some Aspects of Statistical Physics of Correlated Systems*, Ph.D. Thesis, École normale supérieure de Lyon, Lyon (2005).
169. B. Portelli, P. C. W. Holdsworth, M. Sellitto, and S. T. Bramwell, *Phys. Rev.* **E64**, 036111 (2001).
170. K. M. Leung, *Phys. Rev.* **B35**, 226 (1982).
171. G. Ben Arous, L. B. Bogachev, S. A. Molchanov, *Probab. Theory Relat. Fields* **132**, 579 (2005).