Temperature Can Enhance Coherent Oscillations at a Landau-Zener Transition

Robert S. Whitney,[1,2] Maxime Clusel,[3,2] and Timothy Ziman[1,2]

[1]*Laboratoire de Physique et Modélisation des Milieux Condensés (UMR 5493), Université Joseph Fourier and CNRS, Maison des Magistères, BP 166, 38042 Grenoble, France*

[2]*Institut Laue-Langevin, 6 rue Jules Horowitz, BP 156, 38042 Grenoble, France*

[3]*Laboratoire Charles Coulomb, UMR 5221, CNRS and Université Montpellier 2, Montpellier, France*

(Received 1 April 2011; revised manuscript received 27 June 2011; published 15 November 2011)

We consider sweeping a system through a Landau-Zener avoided crossing, when that system is also coupled to an environment or noise. Unsurprisingly, we find that decoherence suppresses the coherent oscillations of quantum superpositions of system states, as superpositions decohere into mixed states. However, we also find an effect we call "Lamb-assisted coherent oscillations," in which a Lamb shift exponentially enhances the coherent-oscillation amplitude. This dominates for high-frequency environments such as super-Ohmic environments, where the coherent oscillations can grow exponentially as either the environment coupling or temperature are increased. The effect could be used as an experimental probe for high-frequency environments in such systems as molecular magnets, solid-state qubits, spin-polarized gases (neutrons or He3), or Bose condensates.

DOI: 10.1103/PhysRevLett.107.210402 PACS numbers: 03.65.Yz, 67.30.ep, 75.50.Xx, 85.25.Cp

Introduction.—A central aspect of quantum systems is that they can be in a coherent superposition of different eigenstates, with observables then undergoing coherent oscillations. One way to create such superpositions is to take a system in its ground state and change its Hamiltonian too rapidly for the system to adiabatically follow the ground state. To distinguish between superpositions and incoherent mixtures, one needs to detect the relatively fast coherent oscillations. These are now detectable in many systems, including superconducting qubits [1], Bose condensates [2,3], polarized He3 [4] or neutrons [5], and probably molecular magnets [6].

All quantum systems have some coupling to degrees of freedom in their environment. Generally, this coupling suppresses coherent oscillations as quantum superpositions decay into incoherent mixtures. The decay mechanism, called decoherence, is usually stronger at higher environment temperature [7]. In this Letter, we ask if this is always the case, by examining the archetypal example of the Landau-Zener transition, which generates a superposition of the ground and an excited state. Various models of the environment will be considered: in some, it behaves as a classical noise field, while in others its quantum nature is taken into account.

We will arrive at the surprising conclusion that the environment can exponentially enhance the coherent oscillations generated at a Landau-Zener transition. This occurs because it modifies the coherent oscillations in two ways. The first is the standard decoherence mechanism, responsible for level-broadening, which suppresses the oscillations [7]. The second is a Lamb shift of the levels [8] which can exponentially reduce or enhance the oscillations. To illustrate these effects, we consider three types of environment: Markovian environments (which we will see exhibit decoherence only), high-frequency environments (which will exhibit Lamb shift only), and Caldeira-Leggett sub- and super-Ohmic environments (which will exhibit both decoherence and Lamb shift). The last will show multiple regimes, due to competition between the two effects.

These noise-enhanced oscillations may remind one of quantum stochastic resonances (QSR) [9,10]; however, there are crucial differences. We have free coherent oscillations at a frequency given by the level-splitting, while QSR's driven oscillations are at the drive's frequency. QSR is typically an enhancement going like a power law of dissipative rates, with some nonexponential modification in those cases where the noise is colored (and thus induces a Lamb shift) [10]. In contrast, the free oscillations we discuss are exponentially enhanced by the stochasticity due to an interplay of a Lamb shift and a Landau-Zener transition.

Model.—The Hamiltonian for the Landau-Zener transition of a two-level system is $\mathcal{H}_t^{\mathrm{LZ}} = -\frac{1}{2}(\nu t \boldsymbol{\sigma}_z + \Delta \boldsymbol{\sigma}_x)$ when written in terms of Pauli matrices. The avoided crossing occurs at time $t = 0$, has width Δ, and is swept through at rate ν. We consider this system coupled to an environment (Fig. 1), with the Hamiltonian

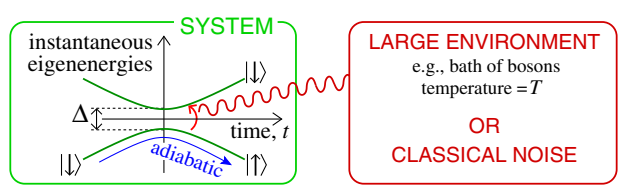

FIG. 1 (color online). We ask how coupling to an environment (with a given spectrum) affects the coherent oscillations of a system swept through a Landau-Zener transition.

$$\mathcal{H}_t^{\text{sys\&env}} = \mathcal{H}_t^{\text{LZ}} - \tfrac{1}{2}\boldsymbol{\sigma}_z \boldsymbol{X} + \mathcal{H}^{\text{env}}, \tag{1}$$

where the environment operator $\boldsymbol{X}$ acts weakly on a huge number of environment modes. The environment Hamiltonian, $\mathcal{H}^{\text{env}}$, is such that these modes have a broad, effectively continuous spread of frequencies.

Such models are well-studied for classical noise [11,12] and quantum environments (addressed using approximate [13–16], exact [17], or numerical methods [18,19]). However, they focus on the transition probability given by $\langle \boldsymbol{\sigma}_z(t)\rangle$. Now that one can measure $s_j(t) = \langle \boldsymbol{\sigma}_j(t)\rangle$ for $j = x, y, z$ in various systems [1,2,4–6], we emphasize that they give us much more information than $s_z(t)$ alone. In this Letter, we consider the magnitude, $A_\perp$, of the coherent oscillations, found by writing $s_y(t) = \frac{1}{2}A_\perp \sin\Phi_t$ for large times t (Φ_t being the oscillation phase). Without an environment, Zener gives

$$A_\perp^0[\Delta] = 4e^{-\pi\Delta^2/(4\nu)} \tag{2}$$

for a near adiabatic transition ($\Delta^2 \gg \nu$) [20]. We will show how the environment could modify this relation.

Master equation.—If the system-environment coupling in $\mathcal{H}_t^{\text{sys\&env}}$ is weak enough to treat in a "golden-rule" manner, then the spin's density matrix obeys the master equation $\dot{\boldsymbol{\rho}}_t = -i[\mathcal{H}_t^{\text{LZ}}, \boldsymbol{\rho}_t]_- - \frac{1}{4}[\boldsymbol{\sigma}_z, (\Xi_t\boldsymbol{\rho}_t - \boldsymbol{\rho}_t\Xi_t^\dagger)]_-$, where the dot denotes a time derivative [21]. The spin operator $\Xi_t = \Xi_t^{\text{R}} + i\Xi_t^{\text{I}}$ is defined by $\Xi_t^{\text{R,I}} = \int_0^\infty dt_1 \alpha^{\text{R,I}}(t_1)\mathcal{U}_{t,t_1}\boldsymbol{\sigma}_z\mathcal{U}_{t,t_1}^{-1}$, where $\mathcal{U}_{t,t_1}$ is the evolution under $\mathcal{H}_\tau^{\text{LZ}}$ from time $(t - t_1)$ to time t. We have split the environment correlation function $\alpha(t) = \langle e^{i\mathcal{H}^{\text{env}}t}\boldsymbol{X}e^{-i\mathcal{H}^{\text{env}}t}\boldsymbol{X}\rangle$ into its real and imaginary parts, $\alpha_{\text{R}}(t)$ and $\alpha_{\text{I}}(t)$, because they are the Fourier transforms of the environment's symmetric and antisymmetric spectral densities, $\mathcal{S}(\Omega)$ and $\mathcal{A}(\Omega)$. Any environment in equilibrium at temperature T has $\mathcal{A}(\Omega) = \mathcal{S}(\Omega) \times \tanh[\Omega/(2k_BT)]$ [22]. This master equation includes weak memory effects; it only reduces to Lindblad's Markovian case if $\alpha(t)$ is a δ function [22].

We parametrize $\boldsymbol{\rho}_t = \frac{1}{2}(1 + s_x\boldsymbol{\sigma}_x + s_y\boldsymbol{\sigma}_y + s_z\boldsymbol{\sigma}_z)$, so that $\boldsymbol{s} = (s_x, s_y, s_z)$ is the spin-polarization vector, and define $\boldsymbol{B}_t = (\Delta, 0, \nu t)$ and $\boldsymbol{e}_z = (0, 0, 1)$, leading to

$$\dot{\boldsymbol{s}} = \boldsymbol{B}_t \times \boldsymbol{s} + \boldsymbol{e}_z \times [\boldsymbol{\xi}^{\text{R}}(t) \times \boldsymbol{s} + \boldsymbol{\xi}^{\text{I}}(t)], \tag{3}$$

where $\xi_j^{\text{R,I}}(t)$ are the $\boldsymbol{\sigma}_j$ components of $\Xi_t^{\text{R,I}}$.

Alternatively, Eq. (3) describes the noise-averaged evolution under the Hamiltonian $\mathcal{H}_t^{\text{LZ}} - \frac{1}{2}X_t\boldsymbol{\sigma}_z$, with the noise field X_t treated using the golden rule [24], with $\overline{X_t} = 0$ and $\overline{X_tX_{t'}} = \alpha_{\text{R}}(t - t')$. Then, $\mathcal{S}(\Omega)$ is the noise power at frequency Ω, while $\mathcal{A}(\Omega) = 0$.

The correlation function $\alpha(t)$ typically decays on a time scale Ω_{m}^{-1}, where Ω_{m} is the characteristic frequency of $\mathcal{S}(\Omega)$ and $\mathcal{A}(\Omega)$. We assume sufficiently fast decay ($\Omega_{\text{m}} \gg \Delta, \nu^{1/2}$) that $\mathcal{U}_{t,t_1} \simeq \exp[i\mathcal{H}_t t_1]$ for all relevant t_1 in Ξ_t. Decoherence is due to $\xi_z^{\text{R}} = [\nu^2t^2\mathcal{S}(0) + \Delta^2\mathcal{S}(B_t)]/(2B_t^2)$ and $\xi_x^{\text{R}} = \Delta\nu t[\mathcal{S}(0) - \mathcal{S}(B_t)]/(2B_t^2)$, for $B_t = |\boldsymbol{B}_t|$. If these are much smaller than B_t, the decoherence rate $T_2^{-1} = [(\Delta^2 + 2\nu^2t^2)\xi_z^{\text{R}} + \Delta\nu t\xi_x^{\text{R}}]/(2B_t^2)$ [25]. The Lamb shift is due to ξ_y^{R}. Defining γ as the relative gap reduction due to this Lamb shift, we have

$$\gamma \equiv -\frac{\xi_y^{\text{R}}}{\Delta} = \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\frac{\mathcal{S}(\Omega)}{\Omega^2 - B_t^2}, \tag{4}$$

while $\xi_x^{\text{I}} = \Delta\nu t\int_{-\infty}^{\infty}(d\Omega/2\pi)\mathcal{A}(\Omega)/(\Omega^3 - B_t^2\Omega)$ and $\xi_y^{\text{I}} = \Delta\mathcal{A}(B_t)/(2B_t)$. In what follows, the t dependence of γ is in terms subdominant in (B_t/Ω_{m}), which we neglect.

Markovian evolution.—We start with classical white noise, with a noise spectrum $\mathcal{S}(\Omega)$ that is Ω-independent [while $\mathcal{A}(\Omega) = 0$]. The noise correlation function $\alpha(t) = a\delta(t)$ corresponds to a complete absence of memory. All ξ's in Eq. (3) are zero except $\xi_z^{\text{R}} = \frac{1}{2}\mathcal{S}(0)$, so there is decoherence but no Lamb shift. This gives the evolution in Fig. 2(a), with noise suppressing the oscillations.

High-frequency noise or environment.—Consider classical noise with $\mathcal{S}(\Omega)$ at much higher frequencies than B_t. Only ξ_y^{R} is nonzero, so there is a Lamb shift but no decoherence. Equation (3) reduces to

$$\begin{pmatrix}\dot{s}_x\\ \dot{s}_y\\ \dot{s}_z\end{pmatrix} = \begin{pmatrix}0 & \nu t & 0\\ -\nu t & 0 & \Delta(1-\gamma)\\ 0 & -\Delta & 0\end{pmatrix}\begin{pmatrix}s_x\\ s_y\\ s_z\end{pmatrix}, \tag{5}$$

whose evolution is shown in Fig. 2(b). The relative gap reduction $\gamma = c\mathcal{S}(\Omega_{\text{m}})/\Omega_{\text{m}}$, with constant $c \sim \mathcal{O}[1]$.

For a quantum environment with $\mathcal{S}(\Omega)$ at much higher frequencies than B_t [as in Fig. 3(a)], only ξ_y^{R} and ξ_x^{I} are nonzero. For large Ω_{m}, we also have $\xi_x^{\text{I}} \ll \xi_y^{\text{R}}$. Ignoring ξ_x^{I}, we recover Eq. (5), with γ now depending on the environment temperature, T. For a spin-boson model—with $\boldsymbol{X} = \sum_n C(\boldsymbol{a}_n^\dagger + \boldsymbol{a}_n)$, where $\boldsymbol{a}_n^\dagger$ ($\boldsymbol{a}_n$) is the nth oscillator's creation (annihilation) operator—we have $\mathcal{S}(\Omega) = 2\pi C^2 d(|\Omega|)\coth[|\Omega|/(2k_BT)]$, where $d(\Omega)$ is the oscillator density at Ω. So, γ is an increasing function of T; thus, the coherent-oscillation magnitude, $A_\perp$, grows exponentially with the environment coupling and strongly with its temperature. Exponential growth with T occurs for $\gamma \propto T$ when k_BT is larger than the typical Ω.

Figure 3(a) shows $d(\Omega) = Ne^{-(\Omega/\Omega_{\text{m}}-1)}/(5\Omega_{\text{m}}^3)$ for $\Omega \geq \Omega_{\text{m}}$ and zero elsewhere, with integrated density N. Then, γ is given by $\gamma = (K/\Omega_{\text{m}})g(\frac{2k_BT}{\Omega_{\text{m}}})$, with $K = NC^2/\Omega_{\text{m}}$ and $g(x) = \int_1^\infty d\mu \times e^{-\mu}\coth[\mu/x]$; so, $g(x \ll 1) = 1$ and $g(x \gg 1) \propto x$.

Sub- or super-Ohmic environment.—Here, we find both a Lamb shift and decoherence because all ξ's are finite. We take $\mathcal{S}(\Omega) = K(|\Omega|/\Omega_{\text{m}})^{1+\epsilon}e^{-|\Omega|/\Omega_{\text{m}}} \times \coth[|\Omega|/(2k_BT)]$ with very large Ω_{m}; this is sub-Ohmic for $\epsilon < 0$ and super-Ohmic for $\epsilon > 0$ [26]. Then, $\mathcal{A}(\Omega) = \mathcal{S}(\Omega)\tanh[\Omega/(2k_BT)]$. For a spin-boson model, this requires an oscillator density of

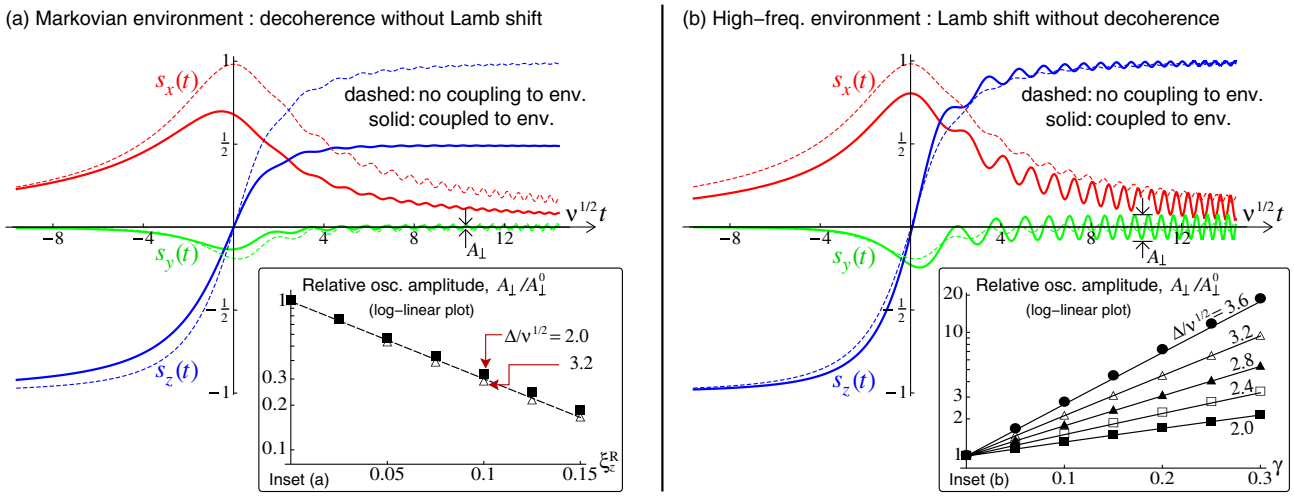

FIG. 2 (color online). The three spin components of the system swept through a Landau-Zener transition, given by numerical evolution of Eq. (3). The main plots in (a) and (b) have $\Delta/\nu^{1/2} = 2.4$ in the presence (solid curves) and absence (dashed curves) of environments. In (a), the environment induces decoherence, reducing the magnitude of the coherent oscillations, while, in (b), the environment induces a Lamb shift, which enhances such oscillations. In (a), the only nonzero ξ is $\xi_z^{\mathrm{R}} = 0.1$. The oscillation magnitude $A_\perp$ is found by fitting $s_y(t)$ for $\nu^{1/2}t$ from 9.5 to 10.5. The inset shows its exponential decay with coupling strength, ξ_z^{R}, at a rate which appears to be Δ-independent. In (b), the only nonzero ξ is $\xi_y^{\mathrm{R}} = -\gamma\Delta$, with $\gamma = 0.3$, for which Eq. (3) reduces to Eq. (5). The inset shows $A_\perp$ growing exponentially with environment coupling; the solid lines are the hypothesis that $A_\perp = (1-\gamma)^{1/2}A_\perp^0[\Delta(1-\gamma)^{1/2}]$, where $A_\perp^0[x]$ is given by Eq. (2). Analysis of oscillations of $s_x(t)$ gives identical results.

$d(\Omega) = N[\Omega_{\mathrm{m}}\Gamma(2+\epsilon)]^{-1}(\Omega/\Omega_{\mathrm{m}})^{1+\epsilon}e^{-\Omega/\Omega_{\mathrm{m}}}$, with $K = NC^2/[\Omega_{\mathrm{m}}\Gamma(2+\epsilon)]$. For $\epsilon < 0$ (sub-Ohmic), the oscillations are suppressed by both decoherence and a Lamb shift which increases the gap. For $\epsilon > 0$ (super-Ohmic), as in Fig. 3(b), the relative gap reduction due to the Lamb shift, γ, has three regimes of behavior:

$$\gamma = \frac{K}{\Omega_{\mathrm{m}}} \times \begin{cases} \mu_0 - \mu_1^{\mathrm{low}} \frac{(k_B T)^{2+\epsilon}}{\Delta^2 \Omega_{\mathrm{m}}^{\epsilon}}, & k_B T \ll \Delta \ll \Omega_{\mathrm{m}}, \\ \mu_0 + \mu_1^{\mathrm{med}} (k_B T/\Omega_{\mathrm{m}})^{\epsilon}, & \Delta \ll k_B T \ll \Omega_{\mathrm{m}}, \\ \mu_1^{\mathrm{high}} k_B T/\Omega_{\mathrm{m}}, & \Delta \ll \Omega_{\mathrm{m}} \ll k_B T. \end{cases} \quad (6)$$

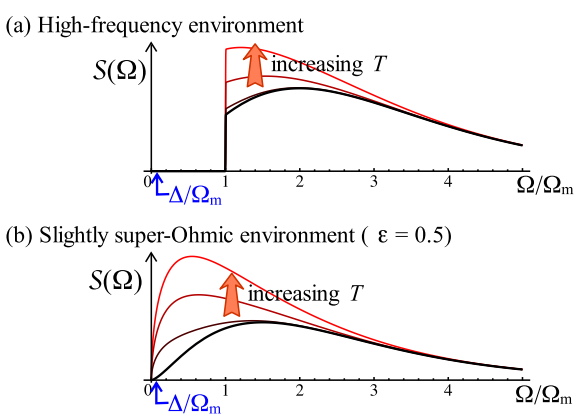

FIG. 3 (color online). Plots of $\mathcal{S}(\Omega)$ showing how weight below $k_B T$ grows significantly as one increases T, while that above only grows slightly; the curves are for $k_B T/\Omega_{\mathrm{m}} = 0$, 1/3, 2/3, and 1. In (a), there is no weight at low Ω, so increasing T enhances γ via the level repulsion discussed in the text. For (b), there is enough weight at low Ω that increasing T reduces γ.

In each regime, we have split the contribution into a dominant T-independent part, μ_0, and a T-dependent part (with prefactor μ_1), subdominant except for high T. All μ's are $\mathcal{O}[1]$; μ_0 and μ_1^{low} are positive for all $\epsilon > 0$, while μ_1^{med} and μ_1^{high} are negative for $\epsilon < 1$ and positive for $\epsilon > 1$. So, if $k_B T \gg \Delta$, γ decays with T for $\epsilon < 1$ and grows for $\epsilon > 1$. At very long times, decoherence suppresses the oscillations. However, for finite times, as in Fig. 2, comparing the Lamb and decoherence effects gives Fig. 4.

Beyond the golden rule.—Using real-time Dyson equations to estimate corrections to the golden rule [22], one finds that Eq. (3) is valid for $A\tau_{\mathrm{corr}} \ll 1$, where τ_{corr} is the time for $\alpha(t)$ to decay and $A = \int_0^\infty dt\alpha(t)$. Thus, the golden rule fails when γ is not small; estimates indicate that oscillations decay in this regime. If $K \ll \Omega_{\mathrm{m}}$, this occurs in the high-T regime at $k_B T_{\mathrm{fail}} \sim \Omega_{\mathrm{m}}^2/K$ (see Fig. 4). If $K \gtrsim \Omega_{\mathrm{m}}$, there is no golden-rule regime and oscillations are always suppressed.

Physical interpretation.—The Lamb shift is due to level repulsion between the spin and the environment modes: high-frequency modes reduce the gap in the spin Hamiltonian, while low-frequency modes enhance it; see Eq. (4). The Landau-Zener transition is exponentially sensitive to this gap; thus, a tiny reduction of it makes the transition much less adiabatic, so coherent oscillations are much larger. In contrast, decoherence comes from environment modes at zero frequency or frequencies in resonance with the spin's level-spacing (see ξ_x^{R} and ξ_z^{R}). As temperature grows, low-frequency modes are more enhanced than high-frequency ones (Fig. 3); their competition causes the T dependences in Fig. 4.

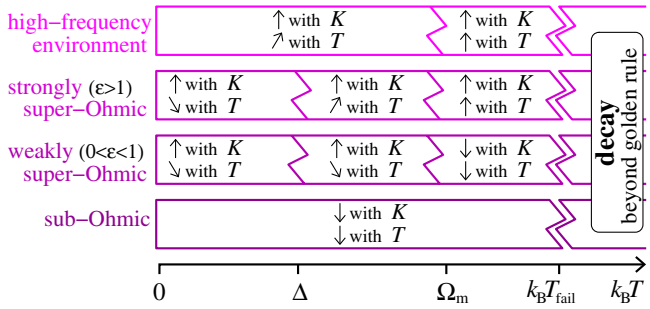

FIG. 4 (color online). Regimes of behavior of the coherent-oscillation magnitude, $A_\perp$. Exponentially strong enhancement with increasing coupling is indicated by "↑ with K" (and exponential reduction by "↓ with K"). Exponentially strong increases and decreases with increasing temperature are indicated by "↑ with T" and "↓ with T," while weaker temperature dependences are indicated by "↗ with T" and "↘ with T."

This picture neglects that the Lamb shift occurs for only one Δ term in Eq. (5), thereby modifying the nature of the dynamics (not just the gap). However, Fig. 2(b) confirms that the picture is qualitatively correct. Thus, we also expect that interference patterns due to multiple passages through an avoided crossing (Landau-Zener-Stückelberg interference) [27–30] will grow exponentially with increasing T whenever the environment is dominated by high frequencies.

Experimental applications.—These "Lamb-assisted coherent oscillations" could be used to probe whether a system has a high-frequency environment (just as spin echo is a probe for low-frequency environments [31]). If the system Hamiltonian is static, then the Lamb shift only gives a weak T dependence to the Larmor precession rate. However, if one sweeps it through a Landau-Zener transition, the coherent-oscillation magnitude becomes exponentially sensitive to this T-dependent shift. A potential application of this probe would be molecular magnets, where there are believed to be two potential sources of relaxation: the primarily high-frequency bath of phonons [32] and the low-frequency bath of nuclear spins. Looking for Lamb-assisted coherent oscillations could clarify which dominates.

One could equally investigate whether high-frequency environments are important sources of dissipation in quantum systems as varied as superconducting qubits [1], Bose condensates [2,3], nanomechanical resonators [30], or spin-polarized gases of He3 [4] or neutrons [5].

[1] M. S. Rudner, A. V. Shytov, L. S. Levitov, D. M. Berns, W. D. Oliver, S. O. Valenzuela, and T. P. Orlando, Phys. Rev. Lett. **101**, 190502 (2008).
[2] T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler, Phys. Rev. Lett. **105**, 204101 (2010); Y.-A. Chen, S. D. Huber, S. Trotzky, I. Bloch, and E. Altman, Nature Phys. **7**, 61 (2010).
[3] D. Witthaut, F. Trimborn, and S. Wimberger, Phys. Rev. Lett. **101**, 200402 (2008).
[4] G. D. Cates, S. R. Schaefer, and W. Happer, Phys. Rev. A **37**, 2877 (1988); A. K. Petukhov, G. Pignol, D. Jullien, and K. H. Andersen, Phys. Rev. Lett. **105**, 170401 (2010).
[5] S. Filipp, J. Klepp, Y. Hasegawa, C. Plonka-Spehr, U. Schmidt, P. Geltenbort, and H. Rauch, Phys. Rev. Lett. **102**, 030404 (2009).
[6] M. Clusel and T. Ziman, arXiv:0705.1631.
[7] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[8] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions* (Wiley, New York, 1992).
[9] T. Wellens, V. Shatokhin, and A. Buchleitner, Rep. Prog. Phys. **67**, 45 (2004); L. Viola, E. M. Fortunato, S. Lloyd, C.-H. Tseng, and D. G. Cory, Phys. Rev. Lett. **84**, 5466 (2000).
[10] M. Grifoni and P. Hänggi, Phys. Rev. Lett. **76**, 1611 (1996); A. N. Omelyanchouk, S. Savel'ev, A. M. Zagoskin, E. Il'ichev, and F. Nori, Phys. Rev. B **80**, 212503 (2009).
[11] Y. Kayanuma, J. Phys. Soc. Jpn. **53**, 108 (1984).
[12] J. I. Vestgården, J. Bergli, and Y. M. Galperin, Phys. Rev. B **77**, 014514 (2008).
[13] Y. Gefen, E. Ben-Jacob, and A. O. Caldeira, Phys. Rev. B **36**, 2770 (1987).
[14] P. Ao and J. Rammer, Phys. Rev. Lett. **62**, 3004 (1989); Phys. Rev. B **43**, 5397 (1991).
[15] E. Shimshoni and A. Stern, Phys. Rev. B **47**, 9523 (1993).
[16] V. L. Pokrovsky and D. Sun, Phys. Rev. B **76**, 024310 (2007).
[17] M. Wubs, K. Saito, S. Kohler, P. Hänggi, and Y. Kayanuma, Phys. Rev. Lett. **97**, 200404 (2006); K. Saito, M. Wubs, S. Kohler, Y. Kayanuma, and P. Hänggi, Phys. Rev. B **75**, 214308 (2007).
[18] P. Nalbach and M. Thorwart, Phys. Rev. Lett. **103**, 220401 (2009).
[19] P. P. Orth, A. Imambekov, and K. Le Hur, Phys. Rev. A **82**, 032118 (2010).
[20] C. Zener, Proc. R. Soc. A **137**, 696 (1932).
[21] This is the Bloch-Redfield equation [22,23], or the weak-coupling limit of the Nakajima-Zwanzig equation [7].
[22] See, e.g., R. S. Whitney, J. Phys. A **41**, 175304 (2008).
[23] Y. Makhlin, G. Schön, and A. Shnirman, Chem. Phys. **296**, 315 (2003).
[24] A. G. Redfield, IBM J. Res. Dev. **1**, 19 (1957).
[25] This gives $T_2^{-1} = \frac{1}{2}\mathcal{S}(0)\cos^2\theta_t + \frac{1}{4}\mathcal{S}(B_t)\sin^2\theta_t$, as expected [23] for an angle θ_t between the z axis and $\boldsymbol{B}_t$.
[26] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).
[27] S. N. Shevchenko, S. Ashhab, and F. Nori, Phys. Rep. **492**, 1 (2010).
[28] W. D. Oliver, Y. Yu, J. C. Lee, K. K. Berggren, L. S. Levitov, and T. P. Orlando, Science **310**, 1653 (2005).

[29] L. Du, M. Wang, and Y. Yu, Phys. Rev. B **82**, 045128 (2010).

[30] M. D. LaHaye, J. Suh, P. M. Echternach, K. C. Schwab, and M. L. Roukes, Nature (London) **459**, 960 (2009).

[31] Y. Nakamura, Yu. A. Pashkin, T. Yamamoto, and J. S. Tsai, Phys. Rev. Lett. **88**, 047901 (2002).

[32] The phonon bath has $\epsilon = 2$, with maximal k vectors giving $\Omega_{\mathrm{m}} \sim 1$ meV and $k_B T \lesssim 1$ meV (i.e., up to a few Kelvin).