Geometrically induced discrete spectrum in curved tubes *

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Abstract

The Dirichlet Laplacian in curved tubes of arbitrary cross-section rotating w.r.t. the Tang frame along infinite curves in Euclidean spaces of arbitrary dimension is investigated. If the reference curve is not straight and its curvatures vanish at infinity, we prove that the essential spectrum as a set coincides with the spectrum of the straight tube of the same cross-section and that the discrete spectrum is not empty.

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1 Introduction

The relationships between the geometric properties of an Euclidean region and the spectrum of the associated Dirichlet Laplacian constitute one of the classical problems of *spectral geometry*, with important motivations coming both from classical and quantum physics. In this paper we consider this type of interactions in the case where the region is an infinite tube. In particular, we are interested in the influence of the curvature on the stability of the essential spectrum and the existence of discrete eigenvalues.

Let $s \mapsto \Gamma(s)$ be a unit-speed infinite curve in $\mathbb{R}^d$, $d \geq 2$. Assuming that the curve is C^d-smooth and possesses an appropriate C^1-smooth Frenet frame $\{e_1, \dots, e_d\}$ (Assumption 1), the i^{th} curvature κ_i of Γ, $i \in \{1, \dots, d-1\}$, is a continuous function of the arc-length parameter $s \in \mathbb{R}$. Given a bounded open connected set ω in $\mathbb{R}^{d-1}$, we define the tube Ω of cross-section ω about Γ by

$$\Omega := \mathcal{L}(\mathbb{R} \times \omega)\,, \qquad \mathcal{L}(s, u_2, \dots, u_d) := \Gamma(s) + u_\mu\, \mathcal{R}_{\mu\nu}(s)\, e_\nu(s)\,, \tag{1}$$

where μ, ν are summation indices taking values in $\{2, \dots, d\}$ and $(\mathcal{R}_{\mu\nu})$ is a family of rotation matrices in $\mathbb{R}^{d-1}$ chosen in such a way that $(s, u_2, \dots, u_d)$ are orthogonal "coordinates" in Ω (*cf* Section 2.2), *i.e.* ω rotates along Γ w.r.t. the Tang frame [15]. We make the hypotheses (Assumption 2) that κ_1 is bounded, $\|\kappa_1\|_\infty \sup_{u \in \omega} |u| < 1$, and Ω does not overlap.

Our object of interest is the Dirichlet Laplacian associated with Ω, *i.e.*

$$-\Delta_D^\Omega \qquad \text{on} \qquad L^2(\Omega). \tag{2}$$

A physical motivation to study this operator for $d = 2, 3$ comes from the fact that it is (up to a physical constant) the quantum Hamiltonian of a free particle constrained to Ω, which is widely used to model the dynamics in mesoscopic systems called *quantum waveguides* [2, 12].

If Γ is a straight line (*i.e.* all $\kappa_i = 0$), then it is easy to see that the spectrum of (2) is purely absolutely continuous and equal to the interval $[\mu_1, \infty)$, where

$$\mu_1 \text{ is the first eigenvalue of the Dirichlet Laplacian in } \omega\,.$$

The purpose of the present paper is to prove that the essential spectrum of the Laplacian (2) is stable as a set under any curvature which vanishes at infinity, and that there is always a geometrically induced spectrum, *i.e.* the spectrum below μ_1, whenever the tube is non-trivially curved.

Theorem 1. *Let Ω be the infinite tube defined above.*

(i) *If* $\lim_{|s| \to \infty} \kappa_1(s) = 0$, *then* $\sigma_{\text{ess}}(-\Delta_D^\Omega) = [\mu_1, \infty)$;

(ii) *If* $\kappa_1 \neq 0$, *then* $\inf \sigma(-\Delta_D^\Omega) < \mu_1$.

Consequently, if the tube is not straight but it is straight asymptotically, then $-\Delta_D^\Omega$ has at least one eigenvalue of finite multiplicity below its essential spectrum, i.e. $\sigma_{\text{disc}}(-\Delta_D^\Omega) \neq \varnothing$.

Here the particularly interesting property is the existence of discrete spectrum, which is a non-trivial property for unbounded regions. From the physical point of view, one then deals with quantum *bound states* of the Hamiltonian (2), which are known to disturb the transport of the particle in the waveguide.

Spectral results of Theorem 1 were proved first by P. Exner and P. Šeba [5] in 1989 for planar strips (*i.e.* $d = 2$) under the additional assumptions that the strip width was sufficiently small and the curvature κ_1 was rapidly decaying at infinity (roughly speaking as $|s|^{-(\frac{3}{2}+\epsilon)}$). A few years later, J. Goldstone and R. L. Jaffe [7] proved the results without the restriction on the strip width, provided the curvature κ_1 had a compact support, and generalised it to the tubes of circular cross-sections in $\mathbb{R}^3$. References to other improvements can be found in the review article [2], where the second part of Theorem 1 was proved under stronger conditions for $d = 2, 3$ and circular cross-section. The first part was proved there just for compactly supported κ_1 (otherwise, under the additional hypothesis that also the first two derivatives of κ_1 vanished at infinity, the authors localised the threshold of the essential spectrum only). Let us also mention the paper [13] where a significant weakening of various regularity assumptions was achieved for $d = 2$. The first part of Theorem 1 was proved for $d = 2$ in the recent paper [10]. The present paper is devoted to a generalisation of the results to higher dimensions and cross-sections rotating along the curve w.r.t. the Tang frame.

Our strategy to prove Theorem 1 is explained briefly as follows. Introducing a diffeomorphism from Ω to the straight tube $\Omega_0 := \mathbb{R} \times \omega$ by means of the mapping $\mathcal{L} : \Omega_0 \to \Omega$, we transfer the (simple) Laplacian (2) on (complicated) Ω into a unitarily equivalent (complicated) operator H of the Laplace-Beltrami form on (simple) Ω_0, *cf* (15). This is the contents of the preliminary Section 2. The rest of the paper, Section 3, is devoted to the proof of Theorem 1. In Section 3.1, we employ a general characterisation of essential spectrum (Lemma 1) adopted from [1] in order to establish the first part of Theorem 1. The reader will notice that the characterisation is better than the classical Weyl criterion in the sense that it deals with quadratic forms instead of the associated operators themselves, *i.e.* we do not need to impose any condition on the derivatives of the coefficients of H in our case. The proof of the second part of Theorem 1 in Section 3.2 is based on the construction of an appropriate trial function for H inspired by the initial idea of [7].

Throughout this paper, we use the repeated indices convention with the range of Latin and Greek indices being $1, \ldots, d$ and $2, \ldots, d$, respectively. The partial derivative w.r.t. a coordinate x_i, $x \equiv (s, u_2, \ldots, u_d) \in \mathbb{R}^d$, is denoted by a comma with the index i.

2 Preliminaries

2.1 The reference curve

Given an integer $d \geq 2$, let $\Gamma : \mathbb{R} \to \mathbb{R}^d$ be a unit-speed C^d-smooth curve satisfying the following hypothesis.

Assumption 1. Γ possesses a positively oriented Frenet frame $\{e_1, \dots, e_d\}$ with the properties that

(i) $e_1 = \dot{\Gamma}$;

(ii) $\forall i \in \{1, \dots, d\}, \quad e_i \in C^1(\mathbb{R}, \mathbb{R}^d)$;

(iii) $\forall i \in \{1, \dots, d-1\}, \ \forall s \in \mathbb{R}, \quad \dot{e}_i(s)$ lies in the span of $e_1(s), \dots, e_{i+1}(s)$.

Remark 1. We refer to [8, Sec. 1.2] for the notion of moving and Frenet frames. A sufficient condition to ensure the existence of the Frenet frame of Assumption 1 is to require that for all $s \in \mathbb{R}$, the vectors $\dot{\Gamma}(s), \Gamma^{(2)}(s), \dots, \Gamma^{(d-1)}(s)$ are linearly independent, *cf* [8, Prop. 1.2.2]. This is always satisfied if $d = 2$. However, we do not assume *a priori* this non-degeneracy condition for $d \geq 3$ because it excludes the curves such that $\Gamma \upharpoonright I$ lies in a lower-dimensional subspace of $\mathbb{R}^d$ for some open $I \subseteq \mathbb{R}$. We also refer to Remark 4 below for further discussions on Assumption 1. □

The properties of $\{e_1, \dots, e_d\}$ summarised in Assumption 1 yield the Serret-Frenet formulae, *cf* [8, Sec. 1.3],

$$\dot{e}_i = \mathcal{K}_{ij}\, e_j \,, \tag{3}$$

where $\mathcal{K}_{ij}$ are coefficients of the skew-symmetric $d \times d$ matrix defined by

$$(\mathcal{K}_{ij}) := \begin{pmatrix} 0 & \kappa_1 & & 0 \\ -\kappa_1 & \ddots & \ddots & \\ & \ddots & \ddots & \kappa_{d-1} \\ 0 & & -\kappa_{d-1} & 0 \end{pmatrix}. \tag{4}$$

Here $\kappa_i : \mathbb{R} \to \mathbb{R}$ is called the i^{th} curvature of Γ. Under Assumption 1, the curvatures are continuous functions of the arc-length parameter $s \in \mathbb{R}$.

2.2 The Tang frame

We introduce now another moving frame along Γ which better reflects the geometry of the curve and will be more convenient for our further purposes.

Let the $(d-1) \times (d-1)$ matrix $(\mathcal{R}_{\mu\nu})$ be defined by the system of differential equations

$$\dot{\mathcal{R}}_{\mu\nu} + \mathcal{R}_{\mu\rho}\, \mathcal{K}_{\rho\nu} = 0 \tag{5}$$

with the initial conditions that $(\mathcal{R}_{\mu\nu}(s_0))$ is a rotation matrix in $\mathbb{R}^{d-1}$ for some $s_0 \in \mathbb{R}$, *i.e.*,

$$\det(\mathcal{R}_{\mu\nu}(s_0)) = 1 \qquad \text{and} \qquad \mathcal{R}_{\mu\rho}(s_0)\,\mathcal{R}_{\nu\rho}(s_0) = \delta_{\mu\nu}. \tag{6}$$

Under our assumptions, the solution of (5) exists and is continuous by standard arguments in the theory of differential equations, *cf* [11, Sec. 4]. Furthermore, the conditions (6) are satisfied for *all* $s_0 \in \mathbb{R}$. Indeed, by means of Liouville's formula [11, Thm. 4.7.1] and $\mathrm{tr}(\mathcal{K}_{\mu\nu}) = 0$, one checks that $\det(\mathcal{R}_{\mu\nu}) = 1$ identically, while the validity of the second condition for all $s_0 \in \mathbb{R}$ is obtained via the skew-symmetry of $(\mathcal{K}_{ij})$:

$$(\mathcal{R}_{\mu\rho}\,\mathcal{R}_{\nu\rho})^{\cdot} = -\mathcal{R}_{\mu\rho}\,\mathcal{R}_{\nu\sigma}\,(\mathcal{K}_{\rho\sigma} + \mathcal{K}_{\sigma\rho}) = 0.$$

We set

$$(\mathcal{R}_{ij}) := \begin{pmatrix} 1 & 0 \\ 0 & (\mathcal{R}_{\mu\nu}) \end{pmatrix} \tag{7}$$

and define the moving frame $\{\tilde{e}_1, \dots, \tilde{e}_d\}$ along Γ by

$$\tilde{e}_i := \mathcal{R}_{ij}\, e_j. \tag{8}$$

Combining (3) with (5) and (4), one easily finds

$$\dot{\tilde{e}}_1 = \kappa_1\, e_2 \qquad \text{and} \qquad \dot{\tilde{e}}_\mu = -\kappa_1\, \mathcal{R}_{\mu 2}\, e_1. \tag{9}$$

We call the moving frame $\{\tilde{e}_1, \dots, \tilde{e}_d\}$ the *Tang frame* throughout this paper because it is a natural generalisation of the Tang frame known from the theory of three-dimensional waveguides [15]. Its advantage will be clear from the subsequent section.

2.3 Tubes

Given a bounded open connected set ω in $\mathbb{R}^{d-1}$, let Ω_0 denote the straight tube $\mathbb{R} \times \omega$. We define the curved tube Ω of the same cross-section ω about Γ as the image of the mapping, *cf* (1),

$$\mathcal{L} : \Omega_0 \to \mathbb{R}^d : \ \{(s,u) \mapsto \Gamma(s) + \tilde{e}_\mu(s)\, u_\mu\}\,, \tag{10}$$

i.e., $\Omega := \mathcal{L}(\Omega_0)$, where $u \equiv (u_2, \dots, u_d)$.

Our strategy to deal with the curved geometry of the tube is to identify Ω with the Riemannian manifold (Ω_0, g_{ij}), where (g_{ij}) is the metric tensor induced by $\mathcal{L}$, *i.e.* $g_{ij} := \mathcal{L}_{,i} \cdot \mathcal{L}_{,j}$, where "$\cdot$" denotes the inner product in $\mathbb{R}^d$. (In other words, we parameterise Ω globally by means of the "coordinates" (s,u).) To this aim, we need to impose a natural restriction on Ω in order to ensure that $\mathcal{L} : \Omega_0 \to \Omega$ is a C^1-diffeomorphism. Namely, defining

$$a := \sup_{u \in \omega} |u|\,, \tag{11}$$

where $|u| := \sqrt{u_\mu u_\mu}$, we make the hypothesis

Assumption 2.

(i) $\kappa_1 \in L^\infty(\mathbb{R})$ and $a\,\|\kappa_1\|_\infty < 1$;

(ii) Ω does not overlap.

Using formulae (9), one easily finds

$$(g_{ij}) = \operatorname{diag}\left(h^2, 1, \dots, 1\right) \qquad \text{with} \qquad h(s,u) := 1 - \kappa_1(s)\,\mathcal{R}_{\mu 2}(s)\,u_\mu\,. \tag{12}$$

By virtue of the inverse function theorem, the mapping $\mathcal{L} : \Omega_0 \to \Omega$ is a local C^1-diffeomorphism provided h does not vanish on Ω_0, which is guaranteed by the condition (i) of Assumption 2 because

$$0 < C_- \le h(s,u) \le C_+ < 2 \qquad \text{with} \quad C_\pm := 1 \pm a\,\|\kappa_1\|_\infty, \tag{13}$$

where we have used that $\mathcal{R}_{\mu 2}\,\mathcal{R}_{\mu 2} = 1$ by (6) and $\sqrt{u_\mu u_\mu} < a$ by (11). The mapping then becomes a global diffeomorphism if it is required to be injective in addition, *cf* the condition (ii) of Assumption 2.

Remark 2. Formally, it is possible to consider (Ω_0, g_{ij}) as an abstract Riemannian manifold where only the curve Γ is embedded in $\mathbb{R}^d$. Then we do not need to assume the condition (ii) of Assumption 2. □

Note that the metric tensor (12) is diagonal due to our special choice of the "transverse" frame $\{\tilde{e}_2, \dots, \tilde{e}_d\}$, which is the advantage of the Tang frame. At the same time, it should be stressed here that while the shape of the tube Ω is not influenced by a special choice of the rotation $(\mathcal{R}_{\mu\nu})$ provided ω is circular, this may not be longer true for a general cross-section. In this paper, we choose rotations determined by the Tang frame due to the technical simplicity.

We set $g := \det(g_{ij}) = h^2$, which defines through $d\mathrm{vol} := h(s,u)\,ds\,du$ the volume element of Ω; here du denotes the $(d-1)$-dimensional Lebesgue measure in ω.

Remark 3 (Low-dimensional examples). When $d = 2$, the cross-section ω is just the interval $(-a, a)$, the curve Γ has only one curvature $\kappa_1 =: \kappa$, the rotation matrix $(\mathcal{R}_{\mu\nu})$ equals (the number) 1 and

$$h(s,u) = 1 - \kappa(s)\,u.$$

When $d = 3$, it is convenient to make the Ansatz

$$(\mathcal{R}_{\mu\nu}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where θ is a real-valued differentiable function. Then it is easy to see that (5) reduces to the differential equation $\dot{\theta} = \tau$, where τ is the torsion of Γ, *i.e.* one put $\kappa := \kappa_1$ and $\tau := \kappa_2$. Choosing θ as an integral of τ, we can write

$$h(s,u) = 1 - \kappa(s)\left(\cos\theta(s)\,u_2 + \sin\theta(s)\,u_3\right).$$

□

Remark 4 (On Assumption 1). As pointed out by P. Exner [3], the existence of the Frenet frame required in Assumption 1 is rather a technical hypothesis only. Indeed, what we actually need is that $\mathcal{L} : \Omega_0 \to \Omega$, with $\mathcal{L}$ given by (10), is a C^1-diffeomorphism, and this is possible to ensure in certain situations even if Assumption 1 does not hold. To see this, let ω be circular and consider a curve Γ possessing the required Frenet frame on $(-\infty, 0)$ and $(0, \infty)$, $e_1 \in C^1(\{0\})$, but $e_\mu \notin C^0(\{0\})$ in the sense that $e_\mu(0+) = S_{\mu\nu} e_\nu(0-)$ for each $\mu \in \{2, \ldots, d\}$, where $(S_{\mu\nu}) \neq 1$ is a constant matrix satisfying relations analogous to (6), *i.e.* the transverse frames $\{e_2(0+), \ldots, e_d(0+)\}$ and $\{e_2(0-), \ldots, e_d(0-)\}$ are rotated to each other (see [14, Chap. 1, pp. 34] for an example of such a curve in $\mathbb{R}^3$). Since the rotation matrix $(R_{\mu\nu})$ is determined uniquely up to a multiplication by a constant rotation matrix, the Tang frame defined by (8) can be chosen to be continuous at zero by the requirement $R_{\mu\nu}(0+) S_{\nu\rho} = R_{\mu\rho}(0-)$. The C^1-continuity at zero then follows by (9) together with the fact that necessarily $\kappa_1(0) = 0$. □

2.4 The Laplacian

Our strategy to investigate the Laplacian (2) is to express it in the coordinates determined by (10). More specifically, using the mapping (10), we identify the Hilbert space $L^2(\Omega)$ with $L^2(\Omega_0, d\mathrm{vol})$ and consider on the latter the sesquilinear form

$$Q(\psi, \phi) := \int_{\Omega_0} \overline{\psi_{,i}}\, g^{ij}\, \phi_{,j}\ d\mathrm{vol}\,, \qquad \psi, \phi \in \mathrm{Dom}\, Q := W_0^{1,2}(\Omega_0, d\mathrm{vol}), \qquad (14)$$

where g^{ij} denotes the coefficients of the inverse of the metric tensor (12). The form Q is clearly densely defined, non-negative, symmetric and closed on its domain. Consequently, there exists a non-negative self-adjoint operator H associated with Q which satisfies $\mathrm{Dom}\, H \subset \mathrm{Dom}\, Q$. We have

$$\mathrm{Dom}\, H = \left\{ \psi \in W_0^{1,2}(\Omega_0, d\mathrm{vol}) \,\middle|\, \partial_i g^{\frac{1}{2}} g^{ij} \partial_j \psi \in L^2(\Omega_0, d\mathrm{vol}) \right\},$$
$$\forall \psi \in \mathrm{Dom}\, H, \quad H\psi = -g^{-\frac{1}{2}} \partial_i \big(g^{\frac{1}{2}} g^{ij} \partial_j \psi \big). \qquad (15)$$

Actually, (15) is a general expression for the Laplace-Beltrami operator in a manifold equipped with a metric (g_{ij}). Using the particular form (12) of our metric tensor, we can write

$$H = -\frac{1}{h}\, \partial_1\, \frac{1}{h}\, \partial_1 - \partial_\mu \partial_\mu + \frac{\kappa_1\, \mathcal{R}_{\mu 2}}{h}\, \partial_\mu \qquad (16)$$

in the form sense.

The norm and the inner product in the Hilbert space $L^2(\Omega_0, d\mathrm{vol})$ will be denoted by $\|\cdot\|_g$ and $(\cdot, \cdot)_g$, respectively. The usual notation without the subscript "g" will be reserved for the similar objects in $L^2(\Omega_0)$.

Remark 5 (Unitarily equivalent operator). Assuming that the reference curve Γ is C^{d+1}-smooth, it is possible to "rewrite" H into a Schrödinger-type operator acting on the Hilbert space $L^2(\Omega_0)$, without the additional weight $g^{\frac{1}{2}}$ in the measure of integration. Indeed, defining $\hat{H} := UHU^{-1}$, where $U : \psi \mapsto g^{\frac{1}{4}}\psi$ is a unitary transformation from $L^2(\Omega_0, d\mathrm{vol})$ to $L^2(\Omega_0)$, we get

$$\hat{H} = -g^{-\frac{1}{4}}\partial_i g^{\frac{1}{2}} g^{ij}\partial_j g^{-\frac{1}{4}} \qquad \text{on} \qquad L^2(\Omega_0)$$

in the form sense. Commuting then $g^{-\frac{1}{4}}$ with the gradient components, we can write

$$\hat{H} = -\partial_1 \frac{1}{h^2}\partial_1 + \partial_\mu\partial_\mu + V \tag{17}$$

in the form sense, where

$$V := -\frac{5}{4}\frac{(h_{,1})^2}{h^4} + \frac{1}{2}\frac{h_{,11}}{h^3} - \frac{1}{4}\frac{h_{,\mu}\,h_{,\mu}}{h^2} + \frac{1}{2}\frac{h_{,\mu\mu}}{h} \tag{18}$$

$$= -\frac{1}{4}\frac{\kappa_1^2}{h^2} + \frac{1}{2}\frac{h_{,11}}{h^3} - \frac{5}{4}\frac{(h_{,1})^2}{h^4}\,. \tag{19}$$

Actually, (17) with (18) is a general formula valid for any C^1-smooth metric of the form $(g_{ij}) = \mathrm{diag}(h^2, 1, \dots, 1)$. (Note that the required regularity is indeed sufficient if the formula for the potential (18) is understood in the weak sense of forms.) In our special case when h is given by (12), we find easily that $h_{,\mu}(\cdot, u) = -\kappa_1\,\mathcal{R}_{\mu 2}$, $h_{,\mu\nu} = 0$, and (19) follows at once. Moreover, (5) gives

$$h_{,1}(\cdot, u) = u_\mu\,\mathcal{R}_{\mu\alpha}\big(\dot{\mathcal{K}}_{\alpha 1} - \mathcal{K}_{\alpha\beta}\mathcal{K}_{\beta 1}\big)\,,$$
$$h_{,11}(\cdot, u) = u_\mu\,\mathcal{R}_{\mu\alpha}\big(\ddot{\mathcal{K}}_{\alpha 1} - \dot{\mathcal{K}}_{\alpha\beta}\mathcal{K}_{\beta 1} - 2\,\mathcal{K}_{\alpha\beta}\dot{\mathcal{K}}_{\beta 1} + \mathcal{K}_{\alpha\beta}\mathcal{K}_{\beta\gamma}\mathcal{K}_{\gamma 1}\big)\,.$$

□

We shall neither need nor use the unitarily equivalent operator from the above remark, however, for motivation purposes, it is interesting to notice that the potential V becomes attractive if a is sufficiently small and the curvatures, together with some of their derivatives, vanish at infinity. Since the latter also implies that h tends to 1 as $a \to 0$, it is easy to see that $\hat{H}$ has always discrete eigenvalues below its essential spectrum for a small enough. In this paper, we prove this property under an asymptotic condition which involves the curvature κ_1 only (*cf* (22) below) and without any restriction on a (except for the natural one in Assumption 2). We also note that various techniques from the theory of Schrödinger operators can be applied to $\hat{H}$, *cf* [2].

2.5 Straight tubes

If the tube is straight in the sense that each $\kappa_i = 0$, then the Laplacian (2) coincides with the decoupled operator

$$H_0 := \overline{-\Delta^{\mathbb{R}} \otimes 1 + 1 \otimes (-\Delta_D^\omega)} \qquad \text{on} \qquad L^2(\mathbb{R}) \otimes L^2(\omega), \tag{20}$$

where 1 denotes the identity operator on the corresponding spaces and the bar stands for the closure. The operators $-\Delta^{\mathbb{R}}$ and $-\Delta_D^\omega$ denote the usual Laplacian on $L^2(\mathbb{R})$ and the Dirichlet Laplacian on $L^2(\omega)$, respectively. Alternatively, H_0 can be introduced as the operator associated with the form Q_0 defined by (14), where now the metric tensor is the identity matrix (δ_{ij}) and $d\mathrm{vol} = ds\,du$ is the Lebesgue measure in $\mathbb{R} \times \omega$.

The operator $-\Delta_D^\omega$ has a purely discrete spectrum consisting of eigenvalues $\mu_1 < \mu_2 \leq \dots \mu_n < \dots$; the corresponding eigenfunctions are denoted as $\mathcal{J}_n$ and we normalise them in such a way that $\|\mathcal{J}_n\|_{L^2(\omega)} = 1$. The lowest eigenvalue μ_1 is, of course, positive, simple and the eigenfunction $\mathcal{J}_1$ can be chosen positive.

In view of the decomposition (20),

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [\mu_1, \infty) \tag{21}$$

and the spectrum is absolutely continuous.

3 Proofs

3.1 The essential spectrum

We prove that the essential spectrum of a curved tube Ω coincides with the one of Ω_0 provided the former is straight asymptotically in the sense that

$$\lim_{|s|\to\infty} \kappa_1(s) = 0. \tag{22}$$

Our method is based on the following characterisation of the essential spectrum of H.

Lemma 1. *$\lambda \in \sigma_{\mathrm{ess}}(H)$ if and only if there exists $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}\,Q$ such that*

(i) $\forall n \in \mathbb{N}, \quad \|\psi_n\|_g = 1,$

(ii) $\forall n \in \mathbb{N}, \quad \mathrm{supp}\,\psi_n \subset \{(s,u) \in \Omega_0 \mid |s| \geq n\},$

(iii) $(H - \lambda)\psi_n \xrightarrow[n\to\infty]{} 0$ *in* $(\mathrm{Dom}\,Q)^*$.

Here $(\mathrm{Dom}\,Q)^*$ denotes the dual of the space $\mathrm{Dom}\,Q$ defined in (14). We note that $H + 1 : \mathrm{Dom}\,Q \to (\mathrm{Dom}\,Q)^*$ is an isomorphism and

$$\|\psi\|_{-1,g} := \|\psi\|_{(\mathrm{Dom}\,Q)^*} = \sup_{\phi\in(\mathrm{Dom}\,Q)\setminus\{0\}} \frac{|(\phi,\psi)_g|}{\|\phi\|_{1,g}} \tag{23}$$

with

$$\|\phi\|_{1,g} := \|\phi\|_{\mathrm{Dom}\,Q} = \sqrt{Q[\phi] + \|\phi\|_g^2}\,.$$

The proof of the above lemma is quite similar to the proof of Lemma 4.2 in [1]. It is based on a general characterisation of essential spectrum, [1, Lemma 4.1], which is better than the Weyl criterion in the sense that the former requires to find a sequence from the form domain of H only (*cf* the statement of Lemma 1 and the required property (iii) with the Weyl criterion [16, Thm. 7.24]). The second property (ii) reflects the fact that the essential spectrum is determined by the geometry at infinity only.

Remark 6. Since the metric (g_{ij}) is uniformly elliptic due to (13), the norms in the spaces $L^2(\Omega_0, d\mathrm{vol})$ and $W_0^{1,2}(\Omega_0, d\mathrm{vol})$ are equivalent with those of $L^2(\Omega_0)$ and $W_0^{1,2}(\Omega_0)$, respectively, and the respective spaces can be identified as sets. In particular,

$$C_-\|\psi\|^2 \leq \|\psi\|_g^2 \leq C_+\|\psi\|^2,$$

and similarly for $\|\cdot\|_{1,g}$ and $\|\cdot\|_{-1,g}$. □

Proof of Theorem 1, part (i). Let $\lambda \in \sigma_{\mathrm{ess}}(H_0) \equiv [\mu_1, \infty)$. By Lemma 1, there exists a sequence $\{\tilde{\psi}_n\}_{n\in\mathbb{N}} \subset \operatorname{Dom} Q_0$ such that it satisfies the properties (i)–(iii) of the Lemma for $g_{ij} = \delta_{ij}$, $H = H_0$ and $Q = Q_0$. We will show that the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ defined by $\psi_n := \tilde{\psi}_n/\|\tilde{\psi}_n\|_g$ for every $n \in \mathbb{N}$ satisfies the properties (i)–(iii) of Lemma 1 for g_{ij}, H and Q, *i.e.* $\sigma_{\mathrm{ess}}(H_0) \subseteq \sigma_{\mathrm{ess}}(H)$. First of all, notice that ψ_n is well defined and belongs to $\operatorname{Dom} Q$ for every $n \in \mathbb{N}$ due to Remark 6. Moreover, writing

$$\tilde{\psi}_n = (1 + H_0)^{-1}(H_0 - \lambda)\tilde{\psi}_n + (1 + H_0)^{-1}(\lambda + 1)\tilde{\psi}_n\,,$$

we see that the sequence $\{\tilde{\psi}_n\}_{n\in\mathbb{N}}$ is bounded in $\operatorname{Dom} Q_0$, and therefore $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\operatorname{Dom} Q$ by Remark 6. Since the conditions (i) and (ii) hold trivially true for $\{\psi_n\}_{n\in\mathbb{N}}$, it remains to check the third one. By the definition of H, *cf* (14), we can write

$$\begin{aligned}&\big(\phi, (H - \lambda)\psi_n\big)_g \\ &\qquad = \big(\phi, (H_0 - \lambda)\psi_n\big) + \big(\phi_{,i}, (g^{\frac{1}{2}} g^{ij} - \delta^{ij})\psi_{n,j}\big) - \lambda\big(\phi, (g^{\frac{1}{2}} - 1)\psi_n\big)\end{aligned}$$

for every $\phi \in \operatorname{Dom} Q$. The Minkowski inequality, the formula (23), the fact that $(H_0 - \lambda)\tilde{\psi}_n \to 0$ in $(\operatorname{Dom} Q_0)^*$ as $n \to \infty$, and a repeated use of Remark 6 yield that it is enough to show that

$$\sup_{\phi\in(\operatorname{Dom} Q)\setminus\{0\}} \frac{\big|\big(\phi_{,i}, (g^{ij} - g^{-\frac{1}{2}}\delta^{ij})\psi_{n,j}\big)_g\big| + \lambda\big|\big(\phi, (1 - g^{-\frac{1}{2}})\psi_n\big)_g\big|}{\|\phi\|_{1,g}} \xrightarrow[n\to\infty]{} 0.$$

However, the latter is easily established by means of the Schwarz inequality, the estimates $\|\phi_{,i}\|_g, \|\phi\|_g \leq \|\phi\|_{1,g}$, the fact that $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\operatorname{Dom} Q$, and the expression for the metric (12) together with (22) and the property (ii) of Lemma 1.

One proves that $\sigma_{\mathrm{ess}}(H) \subseteq \sigma_{\mathrm{ess}}(H_0)$ in the same way. □

Remark 7. It is clear from the previous proof that a stronger result than the first part of Theorem 1 can be proved. If h and $\tilde{h}$ are two positive functions (determining through (12) two tube metrics (g_{ij}) and $(\tilde{g}_{ij})$, respectively) such that $\sup_{u\in\omega} |h(s, u) - \tilde{h}(s, u)| \to 0$ as $|s| \to \infty$, then the essential spectra of the corresponding operators H and $\tilde{H}$ (given by (15) with (g_{ij}) and $(\tilde{g}_{ij})$, respectively) coincide as sets. □

Let us finally notice that a detailed study of the *nature* of the essential spectrum in curved tubes has been performed in [9]; in particular, the absence of singular continuous spectrum is proved there under suitable assumptions about the decay of curvature at infinity.

3.2 The geometrically induced spectrum

In this section we show that $\inf \sigma(H) < \mu_1$ whenever $\kappa_1 \neq 0$, *i.e.* there is always a spectrum below the energy μ_1 in non-trivially curved tubes Ω. We call it geometrically induced spectrum because it does not exist for the straight tube Ω_0, *cf* (21). Furthermore, it follows by the part (i) of Theorem 1 that this geometrically induced spectrum is discrete if we suppose (22) in addition.

Our proof is based on the variational strategy of finding a trial function Ψ from the form domain of H such that

$$Q_1[\Psi] := Q[\Psi] - \mu_1 \, \|\Psi\|_g^2 < 0. \tag{24}$$

The construction of such a Ψ follows the initial idea of [7] and the subsequent improvements of [13] and [2, Thm. 2.1].

Proof of Theorem 1, part (ii). Let $\{\Psi_n\}_{n\in\mathbb{N}} \subset \operatorname{Dom} Q$ and $\Phi \in \operatorname{Dom} Q$. Defining $\Psi_{n,\varepsilon} := \Psi_n + \varepsilon\,\Phi$ for every $(n,\varepsilon) \in \mathbb{N}\times\mathbb{R}$, we can write

$$Q_1[\Psi_{n,\varepsilon}] = Q_1[\Psi_n] + 2\varepsilon\, Q_1(\Phi,\Psi_n) + \varepsilon^2 Q_1[\Phi].$$

Our strategy will be to choose $\{\Psi_n\}_{n\in\mathbb{N}}$ and Φ so that

$$\lim_{n\to\infty} Q_1[\Psi_n] = 0 \qquad \text{and} \qquad \lim_{n\to\infty} Q_1(\Phi,\Psi_n) \neq 0. \tag{25}$$

Then we can choose a sufficiently large $n \in \mathbb{N}$ and a sufficiently small $\varepsilon \in \mathbb{R}$ with a suitable sign so that $Q_1[\Psi_{n,\varepsilon}] < 0$, which proves the claim.

We put $\Psi_n := \varphi_n \otimes \mathcal{J}_1$, where $\mathcal{J}_1$ is the first eigenfunction of $-\Delta_D^\omega$, *cf* Section 2.5, and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a mollifier of 1 in $W^{1,2}(\mathbb{R})$, *i.e.* a family of functions φ_n from $W^{1,2}(\mathbb{R})$ satisfying:

(i) $\forall n \in \mathbb{N}, \quad 0 \le \varphi_n \le 1$,

(ii) $\varphi_n(s) \xrightarrow[n\to\infty]{} 1 \quad$ for a.e. $s \in \mathbb{R}$,

(iii) $\|\dot\varphi_n\|_{L^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0$.

(Probably the simplest example of such a family is given by the continuous even φ_n's such that they are equal to 1 on $[0,n)$, with a constant derivative on $[n,2n+1)$, and equal to 0 on $[2n+1,\infty)$.) Using the expression (16) for the Laplacian and the fact that $(\partial_\mu\partial_\mu + \mu_1)\mathcal{J}_1 = 0$, we obtain immediately that

$$Q_1[\Psi_n] = \left(\Psi_{n,1}, h^{-1}\Psi_{n,1}\right) + \left(\Psi_n, \kappa_1 \mathcal{R}_{\mu 2}\Psi_{n,\mu}\right).$$

The second term at the r.h.s. is equal to zero by an integration by parts, while the first (positive) one can be estimated from above by $C_-^{-1}\|\Psi_{n,1}\|^2$ due to (13). Since $\|\Psi_{n,1}\| = \|\dot{\varphi}_n\|_{L^2(\mathbb{R})}$ by the normalisation of $\mathcal{J}_1$, we verify the first property of (25).

The second property is checked if we take, for instance,

$$\Phi(s,u) := \phi(s)\,\mathcal{R}_{\mu 2}(s)\,u_\mu\,\mathcal{J}_1(u) \ \in \operatorname{Dom} Q,$$

where $\phi \in W^{1,2}(\mathbb{R}) \setminus \{0\}$ is a non-negative function with a compact support contained in an interval where κ_1 is not zero and does not change sign (such an interval surely exists because $\kappa_1 \neq 0$ is continuous). Indeed, in the same way as above, we find

$$Q_1(\Phi,\Psi_n) = \left(\Phi_{,1}, h^{-1}\Psi_{n,1}\right) + \left(\Phi, \kappa_1\mathcal{R}_{\mu 2}\Psi_{n,\mu}\right),$$

where the first term at the r.h.s. tends to zero as $n \to \infty$ because its absolute value can be estimated by $C_-^{-1}\|\Phi_{,1}\|\|\Psi_{n,1}\|$, while the second one is equal to

$$-\tfrac{1}{2}\left(\phi\,\mathcal{J}_1, \kappa_1\mathcal{R}_{\mu 2}\mathcal{R}_{\mu 2}\,\varphi_n\,\mathcal{J}_1\right) = -\tfrac{1}{2}\left(\phi, \kappa_1\varphi_n\right)_{L^2(\mathbb{R})}$$

by an integration by parts; the last identity then holds due to (6) and the normalisation of $\mathcal{J}_1$. Summing up, we conclude that

$$\lim_{n\to\infty} Q_1(\Phi,\Psi_n) = -\frac{1}{2}\int_{\operatorname{supp}\phi} \phi(s)\,\kappa_1(s)\,ds \neq 0$$

by the dominated convergence theorem. □

Remark 8. Suppose Assumptions 1 and 2. If the tube Ω is non-trivially curved and asymptotically straight, it follows by Theorem 1 that $\sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \subset [0,\mu_1)$ and it is not empty. Furthermore, it can be shown by standard arguments (see, *e.g.*, [6, Sec. 8.12]) that the minimum eigenvalue, *i.e.* $\inf\sigma(-\Delta_D^\Omega)$, is simple and has a positive eigenfunction. One also has $\inf\sigma(-\Delta_D^\Omega) > 0$. (Actually, a stronger lower bound to the spectral threshold has been derived in [4].) □

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