A one-parameter refinement of the Razumov-Stroganov correspondence

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Based on joint work with A. Sportiello (Milan University)

Outline

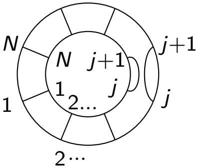
1. The Temperley-Lieb Stochastic process
2. Alternating Sign Matrices or Fully Packed Loop configurations
3. The Razumov Stroganov (ex-)conjecture
4. One parameter refinement of the Razumov–Stroganov correspondence
5. Conclusions & Open Problems

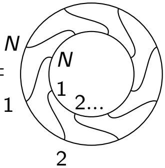
The Temperley-Lieb Stochastic process

Recall the definition of the *Cyclic Temperley-Lieb Algebra* $\mathrm{CTL}_N(\tau)$: free algebra with generators $\{\mathbf{e}_i\}_{i\in\mathbb{Z}}$ and the rotation R

$$\mathbf{e}_i = \mathbf{e}_{i+N} \qquad R^{\pm 1}\mathbf{e}_i = \mathbf{e}_{i\pm 1}R^{\pm 1}$$
$$\mathbf{e}_i = \tau\mathbf{e}_i \qquad \mathbf{e}_i\mathbf{e}_{i\pm 1}\mathbf{e}_i = \mathbf{e}_i$$
$$[\mathbf{e}_i, \mathbf{e}_j] = 0 \qquad |i-j| \neq 1 \pmod{N}.$$

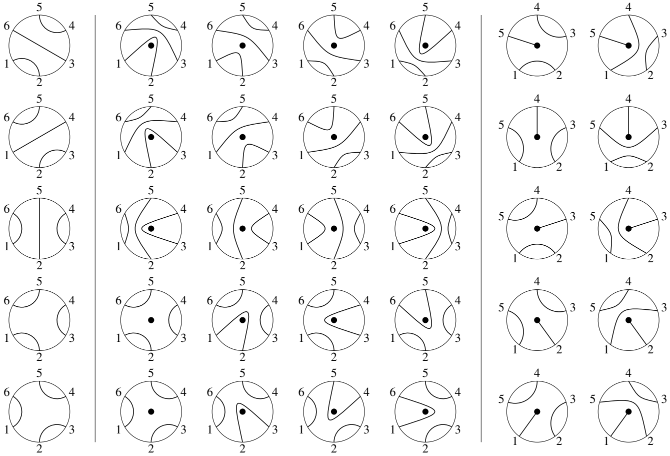
Graphical representation

$\mathbf{e}_j =$ 

$R =$ 

Action on link patterns

We shall be interested in three kind of representations of $\mathrm{CTL}_N(\tau)$ on *link patterns*



LP_{2n} LP^*_{2n} LP^*_{2n+1}

The Temperley-Lieb Stochastic process

For $\tau = 1$ the operator $$H_N = \frac{1}{N}\sum_{i=1}^{N} \mathbf{e}_i$$

is the markov Matrix of the so called Temperley-Lieb Stochastic process [Batchelor, de Gier & Nienhuis, Razumov, Stroganov].

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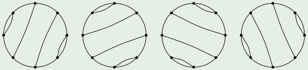
- We are interested in its *Stationary Probability* for the representations LP_{2n} and LP^*_N

$$|\Psi_n\rangle := \sum_{\pi\in\mathrm{LP}_{2n}} \Psi_n(\pi)|\pi\rangle, \qquad |\Psi^*_N\rangle := \sum_{\pi\in\mathrm{LP}^*_N} \Psi^*_N(\pi)|\pi\rangle$$

$$H_N|\Psi\rangle = |\Psi\rangle$$

The Temperley-Lieb Stochastic process

Example (LP_8)



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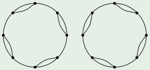

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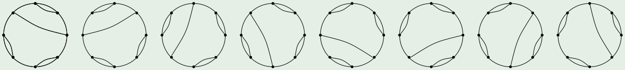

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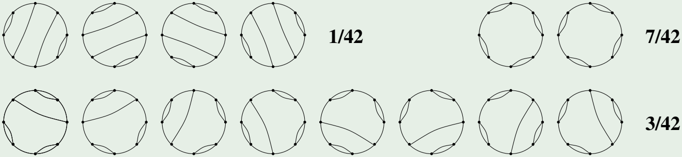

By renormalizing the vectors $|\Psi_n\rangle$ and $|\Psi^*_N\rangle$ [Batchelor, de Gier, Nienhuis, Razumov, Stroganov]

- All the $\Psi_N(\pi)$ are *"small"* integers.

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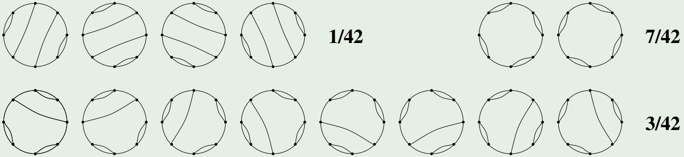

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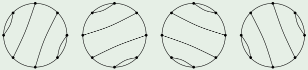

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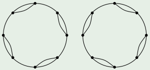

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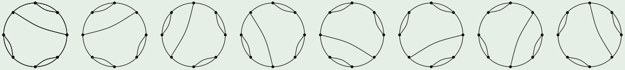

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Does each component have a combinatorial interpretation?

Integrability: inhomogenous Loop Model

R-matrix

$$\hat{X}_i(z) = \frac{qz - q^{-1}}{q - q^{-1}z}\mathbf{1} + \frac{z-1}{q-q^{-1}z}\mathbf{e}_i, \qquad \tau = -q - q^{-1}$$

Yang-Baxter equation

$$\hat{X}_i(z_2)\hat{X}_{i+1}(z_1 z_2)\hat{X}_i(z_1) = \hat{X}_{i+1}(z_1)\hat{X}_i(z_1 z_2)\hat{X}_{i+1}(z_2)$$

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Scattering matrices

$$S_i(\vec{z}) = \hat{X}_{i-2}(z_i/z_{i-1})\hat{X}_{i-3}(z_i/z_{i-2})\cdots\hat{X}_{i+1}(z_i/z_{i+2})\hat{X}_i(z_i/z_{i+1})$$

At $q = e^{2\pi i/3}$ ($\tau = 1$), the scattering equations [Di Francesco, Zinn-Justin]

$$S_i(\vec{z})|\Psi(\vec{z})\rangle = R^{-1}|\Psi(\vec{z})\rangle$$

have a unique solution (up to normalization), polynomial in $\vec{z}$.

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[P. Di Francesco, P. Zinn-Justin]
At $z_i = 1$ the vector $|\Psi(\vec{z})\rangle$ reduces to the stationary probability of the T-L Stochastic model

$$|\Psi_n(\vec{1})\rangle = |\Psi_n\rangle, \qquad |\Psi^*_N(\vec{1})\rangle = |\Psi^*_N\rangle$$

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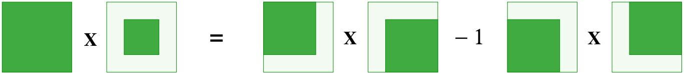
- This allows to derive (directly) determinantal formulae or (through bosonisation of vertex operators) integral formulae for certain components of $|\Psi(\vec{z})\rangle$ or of certain observables.
- Relation with the geometry of *Orbital varieties*.

Alternating Sign Matrices. Why?

Mills, Robbins and Rumsey's extension of Dodgson (aka Lewis Carroll) condensation algorithm ['83]

$$\det\, M \,\det\, M_{1,n}^{1,n} \;=\; \det\, M_n^n \,\det\, M_1^1 \;-1\, \det\, M_1^n \,\det\, M_n^1$$



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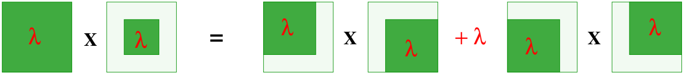

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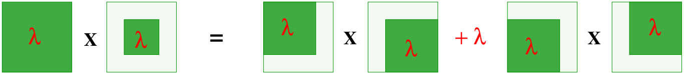

The result is (surprisingly) a Laurent polynomial in entries m_{ij} :

$$\det_\lambda M = \sum_{B \in ASM_n} \lambda^{I(B)} (1+\lambda^{-1})^{N(B)} \prod_{i,j} m_{i,j}^{B_{i,j}}$$

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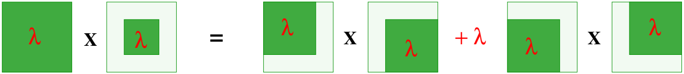

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ASM = Alternating Sign Matrices

. . . The Laurent phenomenon is well known in the context of Hirota equations and has led to the developement of Fomin-Zelevinsky Cluster Algebras

Alternating Sign Matrices [Mills, Robbins, Rumsey]

Square $n \times n$ matrices with entries $0, 1, -1$, such that

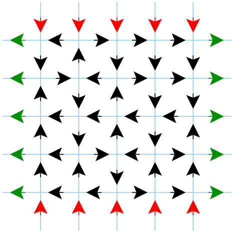
- signs $+1$ and -1 alternate on each row and each column;
- each row and each column sums to 1.

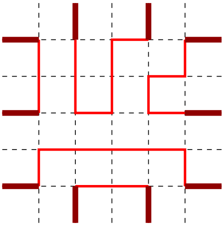
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Enumeration $\quad A_n = \prod_{j=0..n-1} \frac{(3j+1)!}{(n+j)!}$ [Zeilberger '95].

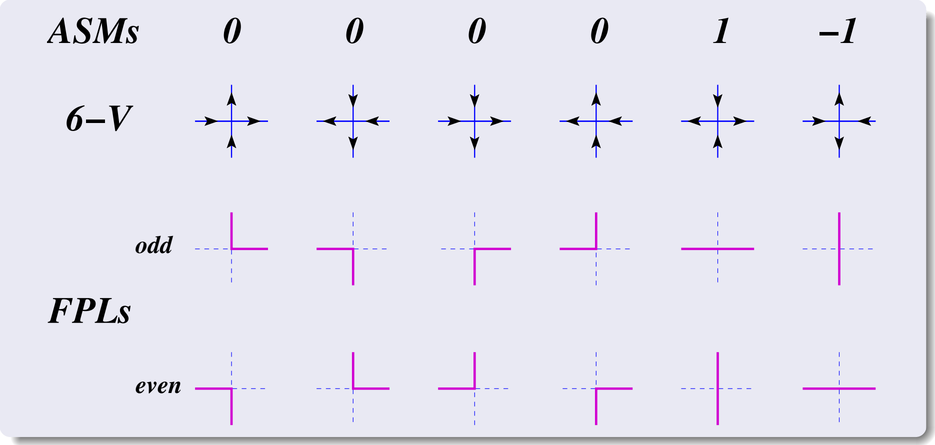
Simpler proof by Kuperberg ['96]: use equivalence with 6-vertex

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



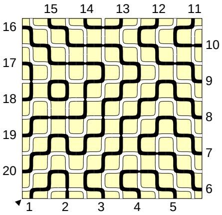


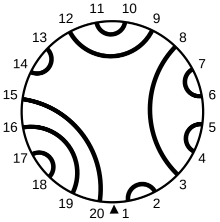
ASM ↔ 6-Vertex (DWBC) ↔ FPL



Refined FPL enumerations

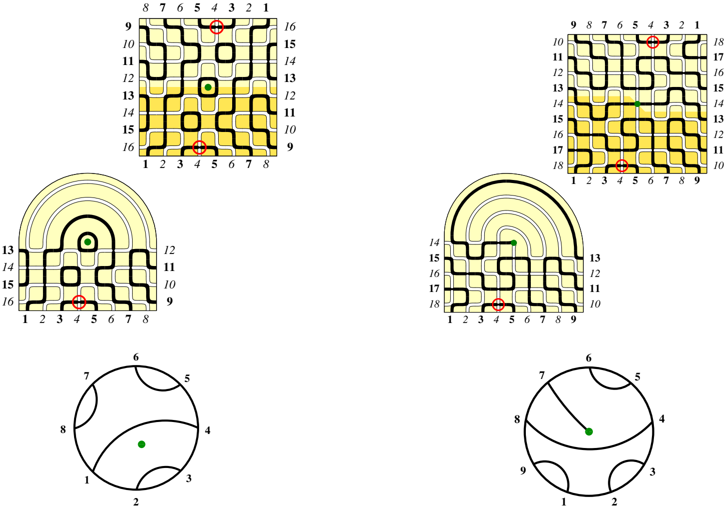
Enumerations of FPLs whose boundary points have a given connection pattern π





$$A_n(\pi) = \#\{\phi \in FPL | \Pi(\phi) = \pi\}$$

Half-Turn ASM



$$A_N^{HT}(\pi) = \#\{\phi \in HTFPL | \Pi(\phi) = \pi\}$$

The Razumov Stroganov ex-conjecture

The number $A_n(\pi)$ are related to stationary probability of the Temperley-Lieb stochastic process

Theorem: R-S ex-conjecture '01 [L.C., A. Sportiello '10]

$$\Psi_n(\pi) = \frac{A_n(\pi)}{\sum_\pi A_n(\pi)}, \qquad \Psi^*_N(\pi) = \frac{A^{HT}_N(\pi)}{\sum_\pi A^{HT}_N(\pi)}.$$

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One can obtain a lot of non trivial informations on the properties of the combinatorial objects on the right-hand side, by studying the left-hand side.

- New proof of the enumeration formula of alternating sign matrices
- Unified proof of enumeration formula of ASM with symmetries
- Number of FPL belonging to certain classes can be counted

Rotational invariance

• A consequence of the RS correspondence is the rotational invariance of the enumerations $A_n(\pi)$ and $A_N^{HT}(\pi)$.

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Theorem (Wieland, '00)

The enumerations $A_n(\pi)$ and $A_N^{HT}(\pi)$ are invariant under cyclic rotations

$$A_n(\pi) = A_n(R \circ \pi), \qquad A_N^{HT}(\pi) = A_N^{HT}(R \circ \pi)$$

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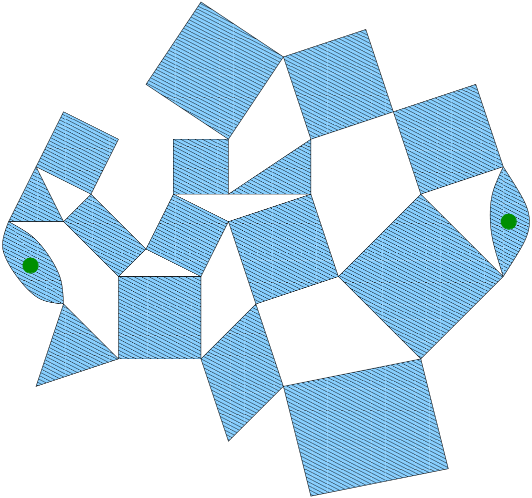
Let's see how the proof works: this will provide a crucial tool for the proof of the RS conjecture.

Rotational invariance: general approach

- Consider a planar graph, which is obtained by gluing at the corners 2-, 3- or 4-gons. Inside a 2-gon we can place a puncture.
- FPL = coloring of the edges such that a vertex of cordination 4 is adjacent to 2 colored edges



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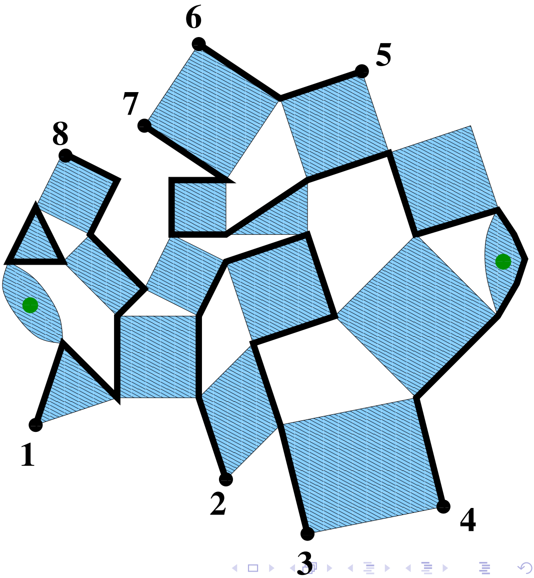
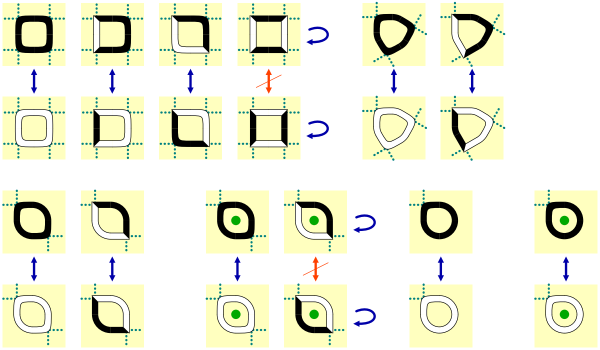
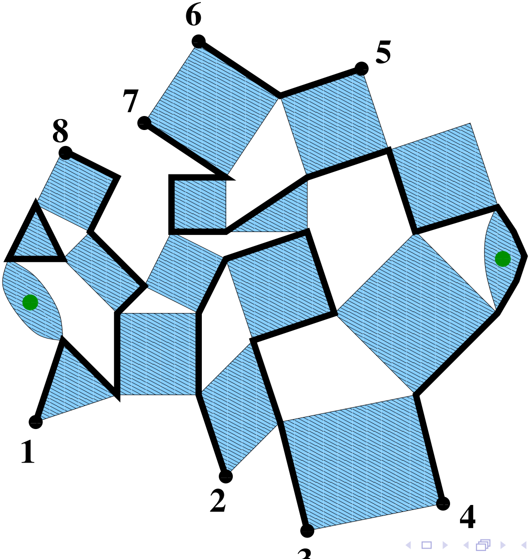


Define the following operation, called Half-Gyration, which preserves the FPL condition at each vertex:



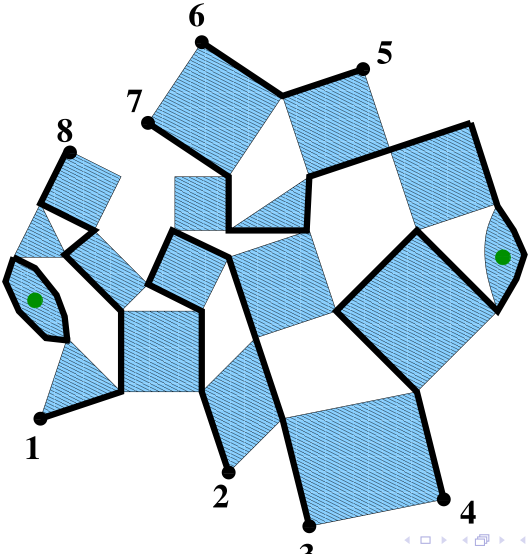
Rotational invariance: general approach 3

Take an FPL



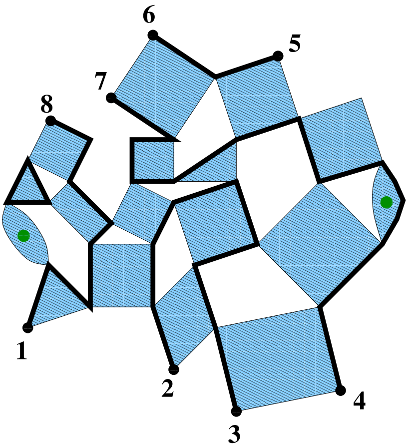
Rotational invariance: general approach 3

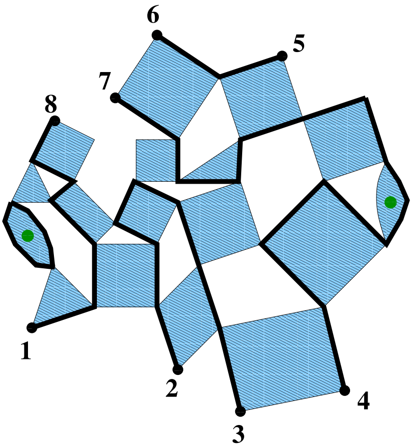
Make a Half-Gyration



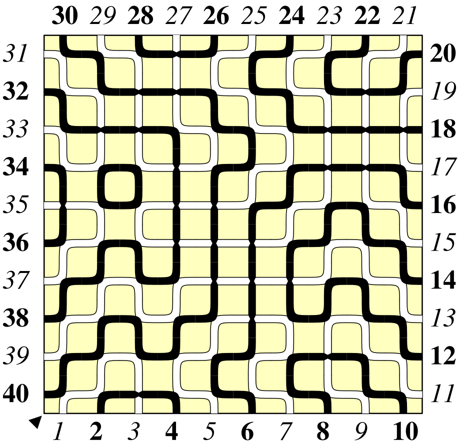
Rotational invariance: general approach 3

The connectivities of the "boundary" points and the "topological" location of the puncture are preserved





Rotational invariance: Wieland's rotation



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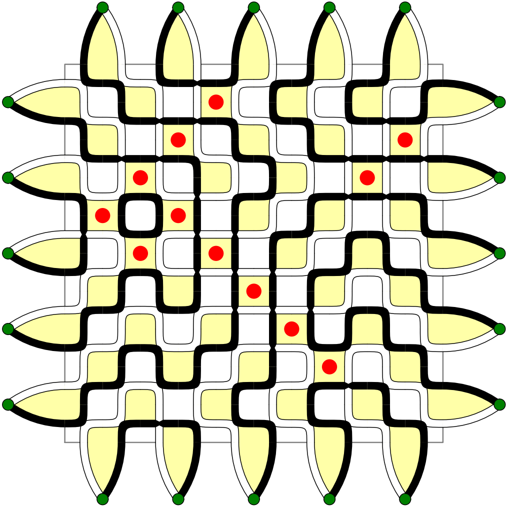

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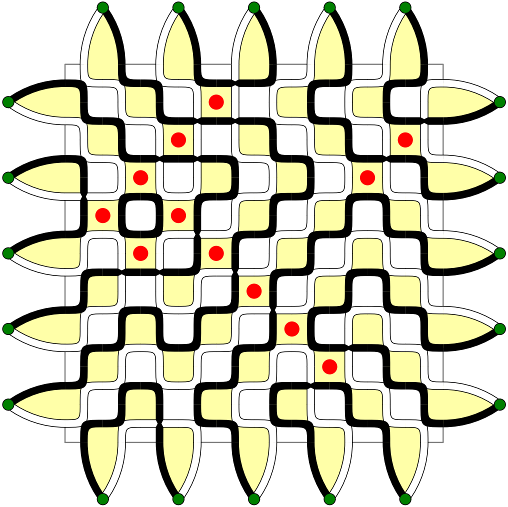

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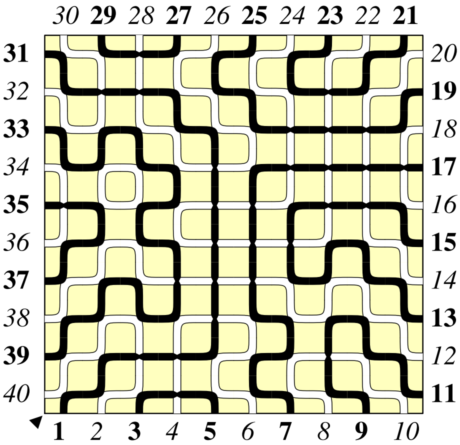

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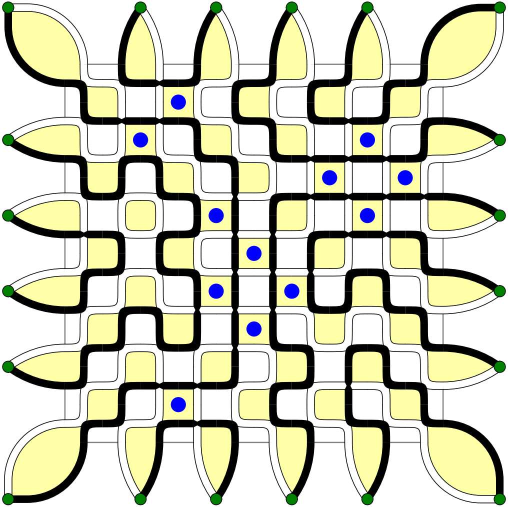

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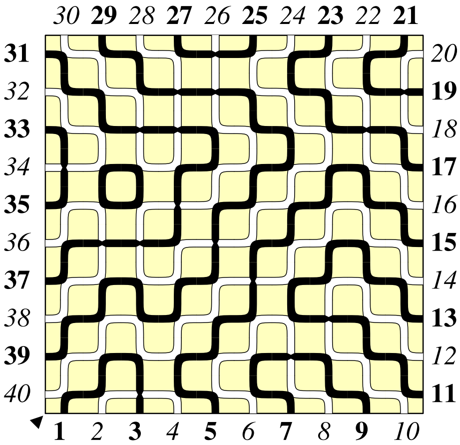

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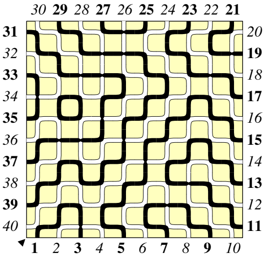

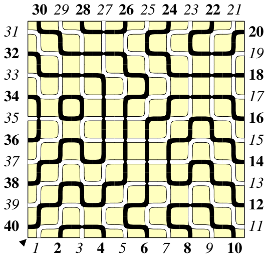

Rotational invariance: Dihedral Domains

The invariance under rotation of the FPL enumerations is valid on more general planar domains that we call *Dihedral Domains*:
- the bulk of the dual graph must be bipartite,
- they have only faces with less than 5 edges
- when joined two consecutives external edges they form a face with less than 4 edges.

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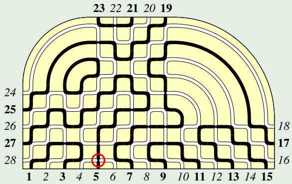

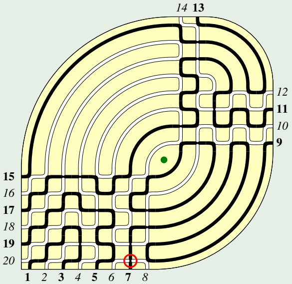

Generalized Razumov-Stroganov correspondence

The Razumov-Stroganov correspondence remains valid on these more general domains!

Define again $A_D(\pi)$ as the number of FPL on the domain D, form the vector

$$|\Psi_D^{FPL}\rangle = \sum_{\pi} A_D(\pi)|\pi\rangle$$

it satisfies [L.C., A. Sportiello, '10]

$$\sum_{i=1}^{2n}(e_i - 1)|\Psi_D^{FPL}\rangle = 0$$

In particular we have proportionality of the enumerations corresponding to the same link pattern on different domains

$$A_D(\pi) = K_D A_n(\pi) \qquad \forall\pi$$

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- The proportionality factor K_D is equal to the number of FPL corresponding to *"Rainbow" Link Patterns*:    and has often an alternative combinatorial interpretation as the number of dimer covering of regions of the hexagonal lattice.
- Alternative way to compute the total enumerations for FPL on several different classes of domains known from Kuperberg and new ones: for example Quarter Turn Symmatric ASMs of size $4n$.

Spectral parameters on the FPL side?

Question

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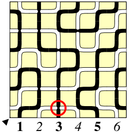
- On the other side Di Francesco noticed that $\Psi(\pi; t) := \Psi(\pi, z_1 = \frac{qt+q^{-1}}{q+q^{-1}t}, \vec{1})$ are polynomials in t with *positive integer coefficients*.

$$\Psi_1(t) = \{1\}$$
$$\Psi_2(t) = \{1, t\}$$
$$\Psi_3(t) = \{1+t, 1, t, t(1+t), t^2\}$$
$$\Psi_4(t) = \{2+3t+2t^2, 1+2t, 1+t+t^2, 2+t, 1, t(2+t), t, t(1+2t), t(2+3t+2t^2), t(1+t+t^2), t^2, t^2(2+t), t^2(1+2t), t^3\}$$

Spectral parameters on the FPL side?

Do these coefficients have a combinatorial meaning?

Di Francesco tried to compare them with enumerations of FPLs weighted by the position $h(\phi)$ of the straight tile on the last row



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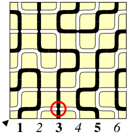

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The weighted enumerations can be gathered in a vector

$$A_n(\pi; t) := \sum_{\phi | \Pi(\phi)=\pi} t^{h(\phi)-1}, \qquad |\Psi_n^{FPL}(t)\rangle := \sum_{\pi} A_n(\pi; t) |\pi\rangle.$$

Unfortunately this doesn't match with $|\Psi_n(t)\rangle$

$$|\Psi_n^{FPL}(t)\rangle \neq |\Psi_n(t)\rangle \quad !!!!$$

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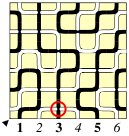

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Conjecture [Di Francesco '04]

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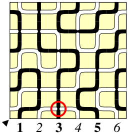

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but it works also on any *Dihedral Domain D*!!!.

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Best possible hope:

• Maybe Di Francesco's way to associate a weight or even of associating a *link pattern* to an FPL is only "almost right".

• There is a new way $\tilde{\pi}(\phi)$ of associating link patterns to FPL such that

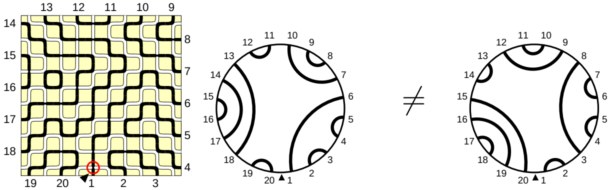
$$|\tilde{\Psi}_D^{FPL}(t)\rangle \propto |\Psi_n(t)\rangle$$

with no need of symmetrization.

The improved refinement

Here is the rule:

If the refinement position is odd, just start the counting of the external points from the refinement position



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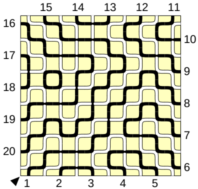

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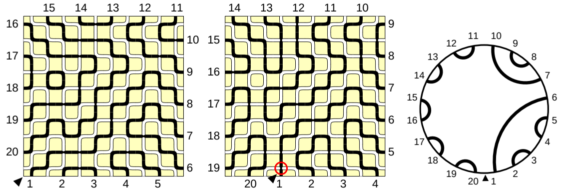

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It is not difficult to show that $|\Psi_n(t)\rangle$ is determined (up to normalization) by the scattering equation

$$(\hat{X}_1(t) - R)|\Psi_n(t)\rangle = 0, \qquad \text{with}$$

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While this theorem is stronger than the Razumov Stroganov original conjecture, its proof is much simpler!!!!
The really non-trivial “work” was in finding the right way to associate a link pattern to an FPL!

The idea of the proof

Apply the two projectors $\mathbf{e}_1$ and $(\mathbf{1}-\mathbf{e}_1)$, to the Scattering equation

$$(\mathbf{1}-\mathbf{e}_1)(t\mathbf{1}-R)|\tilde{\Psi}(t)_D^{FPL}\rangle = 0, \qquad (\mathbf{e}_1 - R\mathbf{e}_N)|\tilde{\Psi}_D^{FPL}(t)\rangle = 0$$

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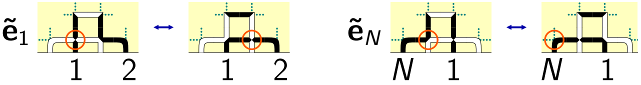
• For the right equation we notice that, while we don't know how to "act" with all the TL generators on a FPL we know how to act with $\mathbf{e}_1$ and $\mathbf{e}_N$



which, combined with two Half-Gyrations, provide the bijection we want.

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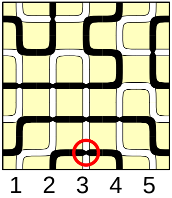

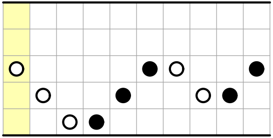

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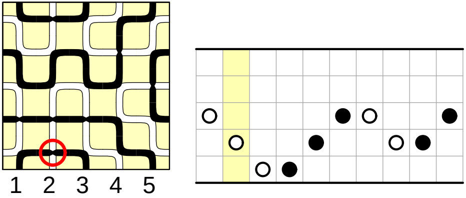

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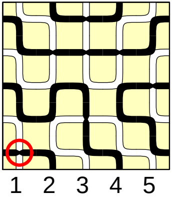

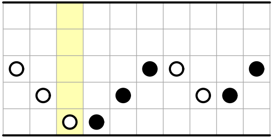

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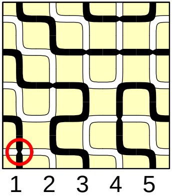

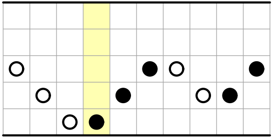

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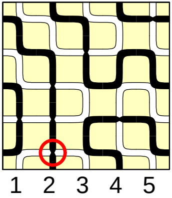

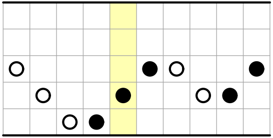

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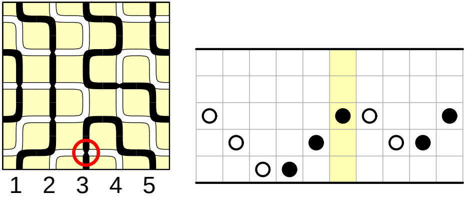

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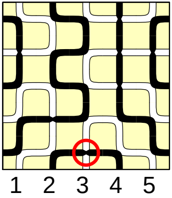

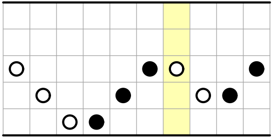

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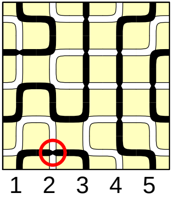

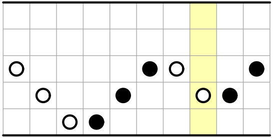

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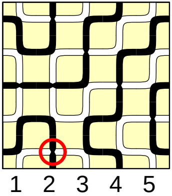

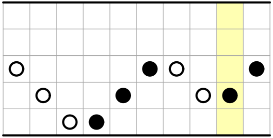

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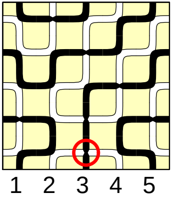

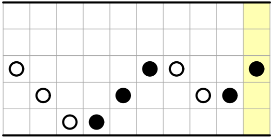

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Will this help in the determination of correspondences with more spectral parameters turned on?

Conclusions & Open problems

At first, we wanted to generalize the proof of the RS conjecture to the refined version by Di Francesco, with one spectral parameter turned on.

At last, we did it, by introducing a new way of associating link patterns to FPL

This leads to a *stronger statement*, and to a new perspective on the Razumov Stroganov correspondence

Will this help in the determination of correspondences with more spectral parameters turned on?

More ambitious: what about the Razumov Stroganov conjectures without cyclic invariance?

Hint for the proof could be to find the class of domains on which these conjectures hold.