Self-Avoiding Walks and Field Theory

Pronob K. Mitter

Département de Physique Théorique, Laboratoire Charles Coulomb
CNRS-Université Montpellier 2

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Outline

Based on :

1. [BGM]: Brydges, Mitter and Guadagni: J Stat Phys (2004) 115: 415-449

2. [BM] : Brydges and Mitter: J Stat Phys (2012), http://arxiv.org/abs/1112.0671

3. [MS] : Mitter and Scoppola: J Stat Phys (2008) 133: 921-1011

4. [BS] : Brydges and Slade: Lecture in International Congress of Mathematicians (2010) and papers in preparation.

Quantum field theory has developed through its interaction with many fields. Besides elementary paricle physics, statistical physics and, in particular, the theory of critical phenomena have played a very important role in the development of Quantum Field Theory. This stimulated the development of Wilson's renormalization group ideas. Random walks and their interactions provide paradigmatic examples of critical phenomena. Field theories arise naturally. By analyzing these theories rigorously we hope to learn something more about both interacting random walks as well as rigorous renormalization group analysis.

Consider two classes of walks on the lattice $\mathbb{Z}^d$: those with nearest neighbour jumps (*simple walks*) and those with long range jumps (*Lévy walks*). An interaction is then added to make them self repelling. The strength of the interaction is kept weak. They are called respectively: (weakly) SAWs and SALWs.

The Green's function of a weakly self avoiding walk on the lattice $\mathbb{Z}^d$ can be represented as the two point correlation function of a special type of supersymmetric field theory. This will be discussed in detail later. In this talk I will discuss the critical limit of a class of weakly self avoiding walks via rigorous renormalization group analysis of the field theory. But first we introduce some basic quantitities for which one wants to have information.

The *Green's function* of a random walk (with or without interactions) from x to y is the expected sum of all walks to go from x to y. For a continuous time walk

$$\mathcal{G}_\mu(x,y) = \int_0^\infty dt\, e^{-\mu t}\, P_t(x,y)$$

where $P_t(x,y)$ is the number of walks from x to y in time t and $\mu \geq \mu_c$ is related to the killing rate. For a free walk $\mu_c = 0$ and P_t is the transition probability. For interactions μ_c has to be fixed so that the walk exists in infinite time. It corresponds to critical temperature in ferromagnetic systems (critical mass in Landau-Ginzburg-Wilson theory).

The next quantitity of interest is the *susceptibility* χ_μ.

$$\chi_\mu = \int_{\mathbb{Z}^d} dx\, G_\mu(x) = \int_0^\infty e^{-\mu t} c(t)$$

$$c(t) = \int_{\mathbb{Z}^d} dx\, p_t(x)$$

The smallest value of μ for which the integral for χ_μ exists defines the critical mass μ_c.

Finally, the end to end distance for a walk starting at the origin is $E(|x_t|)$. The root mean square distance is $\sqrt{E(|x_t|^2)}$. The rms distance does not exist for Lévy walks with long range jumps.

Critical exponents:

The critical exponents η, γ and ν are defined as follows:

In the critical theory, as $|x-y| \to \infty$

$$G_{\mu_c}(x-y) \sim \text{const.}|x-y|^{-(d-\alpha+\eta)}$$

The parameter $\alpha \in (0,2]$ takes the value 2 for simple walks, whereas $\alpha < 2$ for walks with long range jumps.

As $\mu \to \mu_c$ from above

$$\chi_\mu \sim_{\mu \to \mu_c} \text{const.}|\mu - \mu_c|^{-\gamma}$$

In the critical theory ($\mu = \mu_c$) as $t \to \infty$ we have asymptotically

$$E(|x_t|) \sim \text{const.} t^\nu$$

When the underlying random walk is simple, we can replace the mean distance by the root mean square distance.

The same exponent ν arises in another way. For $\mu > \mu_c$, we have asymptotically

$$G_\mu(x-y) \sim e^{-\frac{|x|}{\xi(\mu)}}$$

where the correlation length $\xi(\mu)$ diverges as $\mu \to \mu_c$ like

$$\xi(\mu) \sim_{\mu \to \mu_c} \text{const.}|\mu - \mu_c|^{-\nu}$$

Scaling relation

Just as in ferromagnetic systems we have the Fisher scaling relation

$$\gamma = (2 - \eta)\nu$$

This can be proved by renormalization group arguments at a fixed point at criticality in the supersymmetric field theory representation just as in the Kadanoff argument for ferromagnetic systems.

SAWs: Rigorous results:

$d > 4$: Critical SAWs are diffusive (Brydges-Spencer, [CMP 1984]) for large time for $d > 4$. The proof was by expansion methods (lace expansion). Afterwards Hara and Slade proved (by again using the lace expansion) the existence of the scaling limit in this case. One gets at large distances

$$\mathcal{G}_{\mu_c}(x-y) \sim \ const.|x-y|^{-(d-2)}$$

$$E(x_t^2) \sim \ const.\ t$$

which shows diffusion. The self-repelling interaction is irrelevant.

$d = 4$ is the critical dimension for SAWs. Brydges-Slade use supersymmetric representation plus rigorous RG methods to prove that for critical continuous time SAWs the critical Green's function at large distances behaves as

$$\mathcal{G}_{\mu_c}(x - y) \sim \ const.|x - y|^{-2}$$

This is canonical behaviour in $d = 4$. The fixed point is Gaussian but logarithmic corrections are expected for other quantities.

In d=4, perturbative RG calculations say that at criticality

$$E(x_t^2) \sim \ const.\ t\ log^{\frac{1}{4}} t$$

This has been proved for the hierachical s.a. walk (Brydges-Imbrie) The proof in the general case is part of the program of Brydges and Slade.

In d=3 there are no rigorous results, analogous to the situation for the d=3 Ising model. However Wison- Fisher ϵ expansion in the $n = 0$ limit of the n-vector model (Le Guillou, Zinn-Justin) and direct Monte Carlo studies (Madras-Sokal) indicate non-trivial scaling

$$E(x_t^2) \sim \ const.\ t^{2\nu_{saw}}$$

with $\nu_{saw} = 0.58....$ The Flory argument would have given $\nu_{saw} = \frac{3}{d+2} = \frac{3}{5}$ but this is excluded.
In $d = 2$ we expect that $\nu_{saw} = \frac{3}{4}$ in accord with Flory and this is presumed to be exact but I am not sure that this has been rigorously proved since the scaling limit for $d = 2$ SAWs has not yet been proved to my knowledge.

SALWs:

What is the upper critical dimension? There are two parameters in play: The dimension d and the (Lévy Khintchine) parameter α with $0 < \alpha \leq 2$. $\alpha = 2$ corresponds to simple random walk. For SALWs the upper critical dimension (mean field theory) is $d_c = 2\alpha$. This means that $\epsilon = 2\alpha - d$ can be a small parameter analogous to the ϵ in Wilson and Fisher's ϵ expansion.

The same situation arises for ferromagnets with long range interactions. This was realized long ago by M.Fisher, S-K Ma and B.G. Nickel (1972) who studied the n-vector model with long range interactions and computed critical exponents in low orders of ϵ . By Parisi's perturbation theory argument the $n = 0$ limit would correspond to SALWs.

SALWs: The the idea that $\epsilon = 2\alpha - d > 0$ can taken as a small parameter for SALWs has been exploited by Mitter and Scoppola in ([MS] -2008) for the case $d = 3$ which is below the upper critical dimension $d_c = 3 + \epsilon$. The Green's function of (weakly) self avoiding walks, including those with long range jumps, is expressed as the two point function of a supersymmetric measure. The RG trajectory is then studied rigorously.

SALW: [MS] give a theorem on a global RG trajectory for the critical supersymmetric lattice field theory in $\mathbb{Z}^3$. trajectory for the supermeasure is uniformly bounded on all RG scales and defines a non-Gaussian field theory. The existence of the critical mass and the stable (critical) manifold is proved. The interaction is non-vanishing at all scales. This is the lattice counterpart of a non-Gaussian fixed point in the putative underlying continuum theory.

The proof that the critical manifold exists and that the RG trajectory is uniformly bounded at all scales employs rescaling at each RG step correponding to an assigned field dimension $d_s = \frac{3-\alpha}{2}$. Therefore from general reasoning we expect

$$\mathcal{G}_{\mu_c}(x-y) \sim const.|x-y|^{-(3-\alpha)}$$

In other words $\eta = 0$. But there are other non-trivial critical exponents.

If x_t^α is the continuous time SALW starting at he origin, we want to estimate the critical exponent ν given at criticality $\mu = \mu_c$ by

$$E(|x_t^\alpha|) \sim const.t^\nu$$

Note that we are using the mean distance and not the mean square distance which does not exist for $\alpha < 2$. The mean distance is much harder to estimate than the mean square distance. To estimate ν we can either try to calculate directly the correlation length index or one can adopt the following strategy which uses the scaling relation. In either case we have to stay slightly off critical.

We calculate the susceptibility index γ near criticality directly from the Green's function. An $\epsilon = 2\alpha - d$ expansion for γ in the framework of the renormalization group in the supersymmetric field theory and use of the scaling relation $\gamma = 2\nu$, since $\eta = 0$, then gives

$$\nu = \nu_L \left(1 + \frac{\epsilon}{6} + O(\epsilon^2)\right)$$

$\nu_L = \frac{1}{\alpha}$ is the exponent of the Lévy walk.
Thus we have a non-trivial critical exponent. This is not yet a rigorous estimate. Proving this is a challenging problem.

Hausdorff dimensions (brief review):

Let $E \subset \mathbb{R}^d$ be a bounded subset. Suppose E is covered by open sets $\{E_j\}, 1 \leq j < \infty$ of diameter $|E_j| \leq \epsilon$, $E \subset \cup_{i=1}^{\infty}$. Then for $\beta > 0$, the Hausdorff β measure of E is defined by

$$\mathcal{H}^{\beta}(E) = \lim_{\epsilon \to 0} \inf_{\{E_j\}} \sum_{j=1}^{\infty} |E_j|^{\beta}$$

Then the Hausdorff dimension $d_{\mathcal{H}}(E)$ of E is defined by the common value (the equality below can be proved)

$$d_{\mathcal{H}(E)} = \inf\{\beta : \mathcal{H}^{\beta}(E) = 0\} = \sup\{\beta : \mathcal{H}^{\beta}(E) = \infty\}$$

Random walks can describe bizarre sets (fractal sets) and the Hausdorff dimension need not be an integer! We will see some examples very soon.

One may ask: What is the Hausdorff dimension in the scaling limit of the self avoiding Lévy walk below the critical dimension? This is not known rigorously. But we can make some comments.

Recall first that the typical brownian trajectory $b_t : t \in [0,1]$ has Hausdorff dimension $d_H = 2$ for $d \geq 2$ as first shown by Paul Lévy long ago followed by Taylor and McKean. This turns out to be the inverse of the Brownian exponent $\nu_B = \frac{1}{2}$. For SAWs in $d = 3$ we expect that in the scaling lmit d_H is approximately 1.72..., the inverse of ν_{saw}. For $d = 2$ the Flory estimate is $\nu = \frac{3}{4}$ and this is supported by renormalization group calculations as well as Monte Carlo simulations. We therefore expect that in the scaling limit $d_H = \frac{4}{3}$. The Mandelbrot conjecture says that this is also the Hausdorff dimension of the outer frontier of a typical planar Brownian path. The latter has been proved directly by SLE arguments (Lawler, Schramm and Werner).

For the stable Lévy process $x_t^\alpha : \ 0 < \alpha < 2, \ t \in [0,1]$, Blumenthal and Getoor proved in the 1960s that $d_H = \alpha$ in $d \geq 2$ which is the inverse of ν_L. We conjecture that for the SALW in $d = 3$, $\alpha = \frac{3+\epsilon}{2}$, in the scaling limit the Hausdorf dimension is

$$d_H = \frac{1}{\nu_{SALW}} = \alpha\left(1 - \frac{\epsilon}{6} + \epsilon^2\right)$$

But this requires a proof!

The Lévy Walk

Let $x_t^\alpha \in \mathbb{Z}^d$, $0 < \alpha \leq 2$, be a continuous time Lévy process (called a Lévy walk). This is a compound Poisson process with i.i.d jumps. Jump distribution depends on α. For $\alpha = 2$ we have simple random walks: nearest neighbour jumps. For $\alpha < 2$ the jump distributions have long range tails.

The characteristic function is given by the Lévy Khintchine formula

$$E(e^{i(k,x_t^{(\alpha)})}) = e^{(2d)^{\alpha/2}\, t\, \psi_\alpha(k)}$$

$$\psi_\alpha(k) = -(1 - \frac{1}{d}\sum_{j=1}^{d} \cos\ k_j))^{\alpha/2}$$

$$E(e^{i(k,x_t^{(\alpha)})}) = e^{-t(-\hat{\Delta})(k)^{\alpha/2}}$$

For $\alpha = 2$ the characteristic function is that of a simple random walk. But for $0 < \alpha < 2$ it can be shown that the transition probability

$$P_t^{\alpha}(x, y) \sim \ const. \, |x - y|^{-(d+\alpha)}$$

This means that the variance is infinite but for $\alpha > 1$ the mean distance is finite.

It is easy to show that for $d \leq \alpha$ the walk is recurrent. Thus for a Lévy walk with $1 \leq \alpha < 2$, the walk is recurrent only for $d = 1$. For $d > \alpha$ the walk is transient and the Green's function is

$$G^{\alpha}(x-y) = \int_0^{\infty} dt\, P_t^{(\alpha)}(x,y) = (-\Delta)^{-\alpha/2}(x-y)$$
$$\sim \text{const.}|x-y|^{-(d-\alpha)}$$

when $|x-y| \to \infty$. Later in the RG analysis we will choose $d = 3$ and $\alpha = \frac{3+\epsilon}{2}$, with $0 < \epsilon < 1$.

The weakly Self-Avoiding Walk

Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset. The local time spent by the walk at $x \in \Lambda$ (upto time T) is

$$\tau_T(x) =: \int_0^T ds\, \delta(x_s^\alpha - x)$$

where δ is the lattice delta function. Let dx be the counting measure. Define

$$\tau_T(\Lambda) = \int_\Lambda dx\, \tau_T(x)$$

$$\tau_T^2(\Lambda) = \int_\Lambda dx\, \tau_T(x)^2 = \int_0^T \int_0^T ds\, dt\, \delta^d(x_t - x_s)\, I_{x_t, x_s \in \Lambda}$$

This is a measure of the self intersection.

We define the expectation for a weakly self-avoiding Lévy process by the law

$$E^{(s.a.)}_{g,T,\Lambda}(f(x^{\alpha}_t)) = Z^{-1}\, E\Big(f(x^{\alpha}_t)\, e^{-g\,\tau^2_T(\Lambda)}\Big)$$

Z is a normalization factor and $g > 0$.

The exponential factor makes the walk tend to repel itself (weakly) in Λ for finite $g > 0$.

The Green's function is defined by

$$\mathcal{G}^{\mu}(x,y) = \lim_{\Lambda \uparrow \mathbb{Z}^3} \mathcal{G}^{\mu}_{\Lambda}(x,y)$$

where

$$\mathcal{G}^{\mu}_{\Lambda}(x,y) = \int_0^{\infty} dT\, E_x(e^{-g\tau_T^2(\Lambda)-\mu\tau_T(\Lambda)}\delta^d(x_T^{\alpha}-y))$$

It is asserted that this limit will exist for choice $\mu \geq \mu_c = h(g)$, where $\mu_c = h(g)$ is the critical curve. Proving this and obtaining the asymptotics as $|x-y| \to \infty$ is part of the program.

FIELD THEORY REPRESENTATION:

Let ϕ_1, ϕ_2 be independent identically distributed Gaussian random fields in $\mathbb{Z}^3$ with covariance $\frac{1}{2}C$, and C is the Green's function of the walk.

$\phi = \phi_1 + i\phi_2$ and $\bar{\phi}$ its complex conjugate.

$\psi, \bar{\psi}$ are Grassmann fields (scalar fermions) of degree 1 and -1 respectively.

$$\Phi = (\phi, \psi),\ \bar{\Phi} = (\bar{\phi}, \bar{\psi})$$

Inner product:

$$(\Phi, \Phi) = \Phi\bar{\Phi} = \phi\bar{\phi} + \psi\bar{\psi}$$

Let $\Lambda \subset \mathbb{Z}^3$ be a finite subset. Define

$$V(\Lambda, \Phi, g, \mu) = g \int_\Lambda dx (\Phi\bar{\Phi})^2(x) + \mu \int_\Lambda dx \Phi\bar{\Phi}(x)$$

where the coupling constant $g_0 > 0$ and dx is the counting measure in $\mathbf{Z}^3$. Define the $|\Lambda| \times |\Lambda|$ matrix C_Λ by

$$C_\Lambda(x, y) = C(x - y) : \ x, y \in \Lambda$$

C_Λ is a symmetric, positive definite matrix.

Then our field theory in finite volume Λ is defined by the *supermeasure*

$$d\mu_\Lambda(\Phi) = d\mu_{C_\Lambda}(\Phi) e^{-V_0(\Lambda,\Phi,g,\mu)}$$

where $d\mu_{C_\Lambda}(\Phi)$ is the Gaussian supermeasure

$$d\mu_{C_\Lambda}(\Phi) = \prod_{x\in\Lambda} d\Phi(x)\, e^{-(\Phi, C_\Lambda^{-1}\bar{\Phi})_{L^2(\Lambda)}}$$

$$d\Phi(x) = d\phi_1(x) d\phi_2(x) d\psi(x) d\bar{\psi}(x)$$

Integration over the Grassmann fields is Berezin integration and $d\mu_\Lambda(\Phi)$ is interpreted as a linear functional on the Grassman algebra Ω_Λ (generated by the $\psi, \bar{\psi}$ over the ring of functions which are functionals of the $\phi, \bar{\phi}$). Determinants have cancelled out.

An important fact is that the potential $V_0(\Lambda, \Phi)$ is supersymmetric. Here by *supersymmetry* we mean invariance under the transformation $\mathcal{Q}$ defined on the fields as follows

$$\mathcal{Q}\phi = \psi, \quad \mathcal{Q}\bar{\phi} = -\bar{\psi}$$

$$\mathcal{Q}\psi = \phi, \quad \mathcal{Q}\bar{\psi} = \bar{\phi}$$

This induces in a natural way a supersymmetry transformation $\mathcal{Q}$ on functionals of fields. $\mathcal{Q}$ is nilpotent on gauge invariant functionals F

$$\mathcal{Q}^2 F = 0$$

The supermeasure μ_Λ is $\mathcal{Q}$ invariant. For any functional $F(\Phi)$

$$\mu_\Lambda(\mathcal{Q}F) = 0$$

If in addition $\mathcal{Q}F = 0$ then

$$\mu_\Lambda(F) = F(0)$$

.

As a consequence we have that the supermeasure $d\mu_\Lambda(\Phi)$ is normalized :

$$\int d\mu_\Lambda(\Phi)\, 1 = 1$$

No vacuum energy will be generated.

The McKane-Parisi-Sourlas representation says that:

$$\tilde{\mathcal{G}}^{\mu}_{\Lambda}(x,y) = \int d\mu_{C_\Lambda}(\Phi) e^{-V(\Lambda,\Phi)} \bar{\phi}(x)\phi(y)$$

Proof: (Brydges, Evans and Imbrie and more recently Brydges, Imbrie, Slade).
We want to analyze the supermeasure

$$d\mu_{C_\Lambda}(\Phi) e^{-V(\Lambda,\Phi)}$$

by lattice renormalization group transformations. We will generate a sequence of measures (the RG trajectory) living in smaller and smaller cubes in finer and finer lattices.

A function $C(x, y)$ is said to have finite range L if

$$C(x, y) = 0 : |x - y| \geq L$$

Consider case where C is translation invariant. We are interested in positive definite functions (distributions) C which have expansions

$$C = \sum C_n$$

where each C_n is positive definite, finite range L^n and smooth. Green's functions of self adjoint, second order elliptic operators defined by Dirichlet forms in the continuum or on the lattice have such expansions (Brydges, Guadagni and Mitter, JSP 2004). Fractional powers thereof have such expansions.

Let $\mathcal{X} = \mathbb{R}^d$ or $\mathbb{Z}^d$. Suppose $\phi : \mathcal{X} \to \mathbb{R}$ is a Gaussian random variable distributed according to the covariance C. We have an expansion:

$$\phi = \sum_{n \geq 0} \zeta_n$$

where the ζ_n are independent Gaussian random variables distributed according to C_n. The ζ_n have finite range correlations:

$$E(\zeta_n(x)\zeta_n(y)) = 0 \ : \ |x - y| \geq L^n$$

Let $\Lambda \subset \mathcal{X}$ is a large cube. Let $F(\phi, \Lambda))$ be an L^1 function (on probablity space). Typically to begin with $F = exp - V$ where V is a local functional.

Suppose we want to calculate the expectation $E(F(\phi, \Lambda))$. We write this as a multiple expectation with respect to all the ζ_n, $n = 1, 2, ...$, and carry out the expectations over each n starting with $n = 1$ sequentially. At each step we also perform a rescaling. Together we have a RG step. These steps generalize to super expectations.

Problem: Each such expectation is over fields in a very large region. We have to decouple distant parts to proceed efficiently.

Suppose we have performed $n-1$ RG steps. At this stage we have the fluctuation field ζ_n and the unintegrated fields $\phi = \sum_{j \geq n+1} \zeta_j$. Let F_{n-1} denote the evolved random function.

Pave $\mathcal{X}$ with unit blocks. Λ has the induced paving. Polymers are connected subsets of blocks. A polymer activity is a functional of fields over polymers.

The evolved random function $F_{n-1}(\zeta_n, \phi)$ can be written as a sum of products of activities of disjoint polymers with the spaces in between filled by local functionals of fields independent of ζ_n. Suppose the polymers are sufficiently disjoint. Then the ζ_n expectation factorizes by the finite range property. We are left to study the expectation over a small region. No cluster expansion is necessary.

Lattice RG transformations will be based on finite range expansion of the covariance C, [BGM 2004]. This is an alternative to Kadanoff- Wilson block spin RG.
Let L be a large triadic integer (3^p). Let $\delta_n = L^{-n}$. Let $d_s = \frac{(3-\alpha)}{2}$. We have a sequence of compatible lattices $(\delta_n \mathbb{Z})^3$ and positive definite functions $\Gamma_n : (\delta_n \mathbb{Z})^3 \to \mathbb{R}$ of finite range $\frac{L}{2}$ such that for all $x, y \in (\mathbb{Z})^3$

$$C(x-y) = \sum_{n \geq 0} L^{-2nd_s} \Gamma_n \Big(\frac{x-y}{L^n} \Big)$$

The series converges in $L^\infty(\mathbb{Z}^3)$

Regularity properties

The fluctuation covariances Γ_n have good regularity properties uniform in the lattice scale n. For example

$$\|\partial^m_{\delta_n}\Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,m}$$

Moreover the sequence Γ_n converges exponentially fast to a smooth positive definite continuum function Γ_* of finite range $\frac{L}{2}$ in the following sense: For all $n \geq l \geq d$, with l fixed we have

$$\|\partial^m_{\delta_n}\Gamma_n - \partial^m_c\Gamma_*\|_{L^\infty((\delta_l\mathbb{Z})^3)} \leq c_{L,m}L^{-qn}$$

for some $q > 0$.

Define by recursion the positive definite functions C_n on $(\delta_n \mathbb{Z})^3$

$$C_n(x) = \Gamma_n(x) + L^{-2d_s} C_{n+1}(\frac{x}{L})$$

The C_n have regularity properties uniformly in the lattice scale. They, and their lattice derivatives, converge to smooth continuum functions. Now we can define RG transformations. As usual this is the composition of fluctuation integration (next frame) with rescaling of fields. The scale transformation S_L is defined by $S_L \Phi(x) = L^{-d_s} \Phi(x/L)$.

Let $\Lambda_m = (-L^m/2, L^m/2)^3 \subset \mathbb{R}^3$ and define $\Lambda_{m,n} = \Lambda_m \cap (\delta_n \mathbb{Z})^3$. Start with the density $\mathcal{Z}_0(\Lambda_{N,0}, \Phi) = exp(-V_0(\Lambda_{N,0}, \Phi))$ with initial parameters g_0, μ_0 in a large cube on the unit lattice. There is a sequence of RG transformations which gives the evolution of densities belonging to the Grassman algebra over the ring of bosonic fields on finer and finer lattices

$$T_{N-n,n} \; : \; \Omega^0(\Lambda_{N-n+1,n-1}) \to \Omega^0(\Lambda_{N-n,n})$$

defined by

$$\mathcal{Z}_n(\Lambda_{N-n,n}, \Phi) = \mu_{\Gamma_{n-1}} * \mathcal{Z}_{n-1}(\Lambda_{N-n+1,n-1}, S_L \Phi)$$

$$\int d\mu_{C_0} \mathcal{Z}_0(\Lambda_{N,0}, \Phi) = \int d\mu_{C_n} \mathcal{Z}_n(\Lambda_{N-n,n}, \Phi)$$

At any given step n of the sequence of RG transformations the densities will be given coordinates g_n, μ_n, K_n. Here g_n, μ_n are the evolved parameters of the local potential V_n. and K_n is a so called irrelevant (contracting) term characterized as a polymer activity. The density $\mathcal{Z}_n(\Lambda_{N-n,n}, \Phi)$ can be expressed in terms of these coordinates in a polymer gas representation.

Pave $\mathbb{R}^3$ with a disjoint union of open unit cubes which are integer translates of $(-\frac{1}{2}, \frac{1}{2})^3$. Take their intesection with a fine lattice. These are unit blocks of the fine lattice. A polymer $X \subset \Lambda_{N-n,n}$ is by default a *connected* union of unit blocks. Two disjoint polymers are separated by a distance ≥ 1. A polymer activity K_n is a map $(X, \Phi) \to \Omega^0(X)$ and

$$\mathcal{Z}_n(\Lambda_{N-n,n}) = \sum_{N \geq 0} \frac{1}{N!} e^{-V_n(X_c)} \sum_{X_1,..X_N} \prod_{j=1}^{N} K_n(X_j)$$

The sum is over mutually disjoint connected polymers in $\Lambda_{N-n,n}$.

The density $\mathcal{Z}_n(\Lambda_{N-n,n}, \Phi + \xi)$ can also be written as

$$\mathcal{Z}_n(\Lambda_{N-n,n}) = \sum_{N \geq 0} \frac{1}{N!} e^{-\tilde{V}(Y_c)} \sum_{Y_1,..Y_N} \prod_{j=1}^{N} \mathcal{B}K(Y_j)$$

The sum is now over mutually disjoint connected L- polymers in $\Lambda_{N-n,n}$. $\mathcal{B}K$ is a non-linear functional of K_n, $\tilde{V}_n$ which depends on Φ, ξ. $\tilde{V}_n$ is a yet to be chosen local potential which depends only on Φ . Then the *fluctuation* map $S_{L\mu_{\Gamma_n}*}$ which integrates out the ξ *factorizes* over the product of polymer activities because of the finite range property of Γ_n since the connected L- polymers are separated by a distance $\geq L$. Thus the polymer representation is preserved after fluctuation integration.

The fluctuation inntegration plus rescaling has given a map

$$V_n \to \tilde{V}_{n,L} = S_L V_n, \quad K_n \to \mathcal{F} K_n$$

The image polymer activity lives on polymers in $\Lambda_{N-1-n,n+1}$. The polymers have become finer and the cube smaller.

We can profit from the arbitrariness in the choice of $\tilde{V}_n$ to subtract out the (localized) expanding parts of $\mathcal{F}K_n$, and absorb them in $\tilde{V}_{n,L}$ thus producing a flow of parameters. This subtraction operation on $\mathcal{F}K_n(X, \Phi)$ needs only to be be done for small sets X, because large sets provide contracting contributions. The new subtracted polymer activities have good contraction properties (irrelevant terms). Finally note that supersymmetry is preserved by these maps. No vacuum energy terms are produced by supersymmetry.

This procedure produces our final RG map f_{n+1}

$$f_{n+1,V}(V_n, K_n) = V_{n+1}, \quad f_{n+1,K}(V_n, K_n) = K_{n+1}$$

Using second order perturbation theory,

$$K_n = e^{-V_n} Q_n + R_n$$

Q_n is a second order contribution. It is form invariant and depends on g_n, μ_n and a non-local kernel w_n which converges fast to continuum kernel. We will ignore it for simplicity. R_n is a remainder. Let $u_n = (g_n, \mu_n, R_n)$ represent a point on the RG trajectory. The RG map produces a discrete flow:

$$u_{n+1} = f_{n+1}(u_n)$$

The flow map in components is:

$$g_{n+1} = f_{n+1,g}(u_n) = L^\epsilon g_n((1 - L^\epsilon a_n g_n) + \xi_n(u_n)$$

$$\mu_{n+1} = f_{n+1,\mu}(u_n) = L^{\frac{3+\epsilon}{2}} \mu_n - L^{2\epsilon} b_n g_n^2 + \rho_n(u_n)$$

$$R_{n+1} = f_{n+1,R}(u_n) =: U_{n+1}(u_n)$$

The coefficients a_n are positive and converge fast to a limit $a_c > 0$. We have an approximate flow $\bar{g}_n$ obtained by ignoring the remainder ξ_n and replacing a_n by its limiting value a_c. This approximate flow has an attractive fixed point $\bar{g} = O(\epsilon)$, for ϵ sfficiently small.

Let $\tilde{g}_n = g_n - \bar{g}$. Then $v_n = (\tilde{g}_n, \mu_n, R_n)$ are the new coordinates. Then

$$\tilde{g}_{n+1} = f_{n+1,g}(v_n) = (2 - L^{\epsilon})\tilde{g}_n + \tilde{\xi}_n(v_n)$$

$$\mu_{n+1} = f_{n+1,\mu}(v_n) = L^{\frac{3+\epsilon}{2}}\mu_n + \tilde{\rho}_n(v_n)$$

$$R_{n+1} = f_{n+1,R}(v_n) =: U_{n+1}(v_n)$$

are the new flow equations. $(2 - L^{\epsilon}) = (1 - O(logL)\epsilon$ is a contraction factor.

Banach spaces:

We will consider the RG action on polymer activities in the formal infinite volume limit. This makes sense because of the finite range property of fluctuation covariances.

We endow polymer activities K_n with a norm $\|\cdot\|_n$. This norm tends to a continuum norm as $n \to \infty$. This gives us a Banach space of Polymer activities at scale n. The norm measures:

1. Large ϕ field growth (large field regulator)
2. Partial derivatives in the ϕ, ψ, finite number in ϕ, and all in ψ.
3. Puts in a weight which says that large polymers contribute small amount (large set regulator).

We introduce a second norm $|\cdot|_n$. This is the same as the previous norm except that we evaluate poymer activities at $\phi = 0$, and therefore no large field growth to be measured.

We measure the remainder R_n in a norm $|||\cdot|||_n$, where

$$|||R_n|||_n = max\{|R_n|,\ \bar{g}^2||R_n||\}$$

Define a Banach space E_n consisting of elements $v_n = (\tilde{g}_n, \mu_n, R_n)$ with norm

$$||v_n|| = max\{(\nu\bar{g})^{-1}|\tilde{g}_n|, \bar{g}^{-(2-\delta)}|\mu_n|, \bar{g}^{-(11/4-\eta)}|||R_n|||_n\}$$

where $\delta, \eta > 0$ are very small numbers and $0 < \nu < 1/2$.

Let $E_n(r) \subset E_n$ be an open ball of radius r centered at the origin. Let $v_n \in E_n(1)$. Then our first theorem says

$$|\xi_n(v_n)| \leq C_L \bar{g}^{11/4-\eta}, \quad |\rho_n(v_n)| \leq C_L \bar{g}^{11/4-\eta}$$

These are estimates for the error terms in the g_n, μ_n flow. Moreover $R_{n+1} = U_{n+1}(v_n)$ has the bound

$$|||U_{n+1}(v_n)|||_{n+1} \leq L^{-1/4} \bar{g}^{11/4-\eta}$$

On the right hand side we have a contraction factor. We also have Lipshitz continuity in the above norms with the same constants.

Existence of stable manifold

We consider a Banach **E** space of sequences $\mathbf{s} = \{v_n\}_{n\geq 0}$, with $v_n \in E_n$, supplied with the norm

$$||\mathbf{s}|| = \sup_{n\geq 0} ||v_n||_n$$

$\mathbf{E}(r) \subset \mathbf{E}$ is an open ball of radius r. Let $v_0 = (\tilde{g}_0, \mu_0, 0)$. The next theorem states that if $\tilde{g}_0$ is held in a sufficiently small open ball then there exists a Lipshitz continuous critical mass $\mu_c(\tilde{g}_0)$ such that the trajectory $u_{n+1} = f_{n+1}(u_n)$ is uniformly bounded in the norm on the space of sequences.

Main Theorem:

Let L be sufficiently large, ν be sufficiently small depending on L, then ϵ sufficiently small depending on L. Let $v_0 \in E_0(1/32)$. Let $\tilde{g}_0 \in U_0(r) \subset \mathbb{R}$ where $U_0(r)$ is an open ball of radius r sufficiently small. Then there is a Lipshitz continuous function $h : U_0(r) \to \mathbb{R}$ such that if $\mu_0 = h(\tilde{g}_0)$ then there is a sequence $\mathbf{s} = \{v_n\}_{n \geq 0}$ in $\mathbf{E}(1/4)$ satisfying $v_{n+1} = f_{n+1}(v_n)$ for all $n \geq 0$. The stable manifold is the graph $W_S = \{\tilde{g}_0, h(\tilde{g}_0)\}$.

Corollary: the theorem implies $v_n \in E_n(1/4)$ for all $n \geq 0$ and hence $|\tilde{g}_n| = |g_n - \bar{g}| \leq 1/4\nu\bar{g}$. Since $0 < \nu < 1/2$, it follows that g_n is non-vanishing at all scales.

Proof of the theorem is done in 2 steps inorder to circumvent lattice artifacts.

First we iterate the RG map a large (but finite) number of times. We come to a high scale (sufficiently fine lattice). We are then able to prove the existence of the stable manifold starting at this scale using a fixed point argument on the Banach space of sequences plus an argument of Schub. Exponential convergence of finite range fluctuation covariances to the continuum plays an important role.

Stable manifold has been established at a high scale. Now use the Banach space implicit function theorem to prove that there exists a unique unit lattice critical mass which is a C^1 function of $\tilde{g}_0$ in a small enough neigborhood such that after a finite number of iterations we arrive at the data of the stable manifold at the high scale.