

Exclusion Process and Growth Models

K. Mallick

Institut de Physique Théorique, CEA Saclay (France)

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Introduction

The statistical mechanics of a system at thermal equilibrium is encoded in the **Boltzmann-Gibbs canonical law**:

$$P_{\text{eq}}(\mathcal{C}) = \frac{e^{-E(\mathcal{C})/kT}}{Z}$$

the **Partition Function Z** being related to the Thermodynamic **Free Energy F** :

$$F = -kT \text{Log } Z$$

This provides us with a **well-defined prescription** to analyze systems *at equilibrium*:

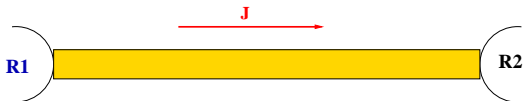
- (i) Observables are mean values w.r.t. the **canonical measure**.
- (ii) Statistical Mechanics predicts **fluctuations** (typically Gaussian) that are out of reach of Classical Thermodynamics.

Systems far from equilibrium

No fundamental theory is yet available.

- What are the **relevant macroscopic parameters**?
- Which **functions** describe the state of a system?
- Do **Universal Laws** exist? Can one define Universality Classes?
- Can one postulate a general form for the **microscopic measure**?
- What do the **fluctuations** look like ('non-gaussianity')?

Example: Stationary driven systems in contact with reservoirs.



Non-vanishing stationary current: a fingerprint for nonequilibrium

Rare Events and Large Deviations

Let $\epsilon_1, \dots, \epsilon_N$ be N independent binary variables, $\epsilon_k = \pm 1$, with probability p (resp. $q = 1 - p$). Their sum is denoted by $S_N = \sum_1^N \epsilon_k$.

- The **Law of Large Numbers** implies that $S_N/N \rightarrow p - q$ a.s.
- The **Central Limit Theorem** implies that $[S_N - N(p - q)]/\sqrt{N}$ converges towards a Gaussian Law.

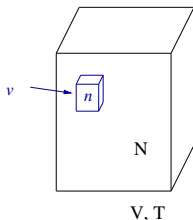
One can show that for $-1 < r < 1$, in the large N limit,

$$\Pr\left(\frac{S_N}{N} = r\right) \sim e^{-N\Phi(r)}$$

where the positive function $\Phi(r)$ vanishes for $r = p - q$. The function $\Phi(r)$ is a **Large Deviation Function**: it encodes the **probability of rare events**.

$$\Phi(r) = \frac{1+r}{2} \ln\left(\frac{1+r}{2p}\right) + \frac{1-r}{2} \ln\left(\frac{1-r}{2q}\right)$$

Density fluctuations in a gas



$$\text{Mean Density } \rho_0 = \frac{N}{V}$$

$$\text{In a volume } v \text{ s. t. } 1 \ll v \ll V$$
$$\left\langle \frac{n}{v} \right\rangle = \rho_0$$

The probability of observing large fluctuations of density in v is given by

$$\text{Pr} \left(\frac{n}{v} = \rho \right) \sim e^{-v \Phi(\rho)}$$

with $\Phi(\rho) = f(\rho, T) - f(\rho_0, T) - (\rho - \rho_0) \frac{\partial f}{\partial \rho_0}$ where $f(\rho, T)$ is the *free energy per unit volume* in units of kT : the **Thermodynamic Free Energy** can be viewed as a **Large Deviation Function**.

Conversely, large deviation functions *may* play the role of potentials in non-equilibrium statistical mechanics.

A Symmetry of the Large Deviation Function

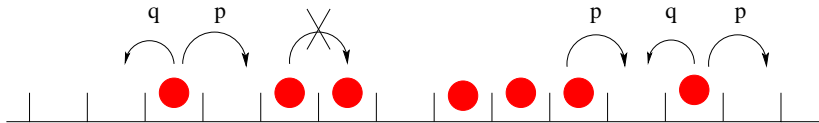
Large deviation functions obey a symmetry that remains valid far from equilibrium:

$$\Phi(r) - \Phi(-r) = Ar$$

The coefficient A is a constant, e.g. $A = \ln q/p$ in the example above.

This **Fluctuation Theorem of Gallavotti and Cohen** is deep and general: it reflects covariance properties under **time-reversal**.

In the vicinity of equilibrium the Fluctuation Theorem yields the fluctuation-dissipation relation (Einstein), Onsager's relations and linear response theory (Kubo).



Asymmetric Exclusion Process. A **paradigm** for non-equilibrium Statistical Mechanics.

- **EXCLUSION:** Hard core-interaction; at most 1 particle per site.
- **ASYMMETRIC:** External driving; breaks detailed-balance (no microreversibility \rightarrow current)
- **PROCESS:** Stochastic Markovian dynamics; no Hamiltonian

ORIGINS

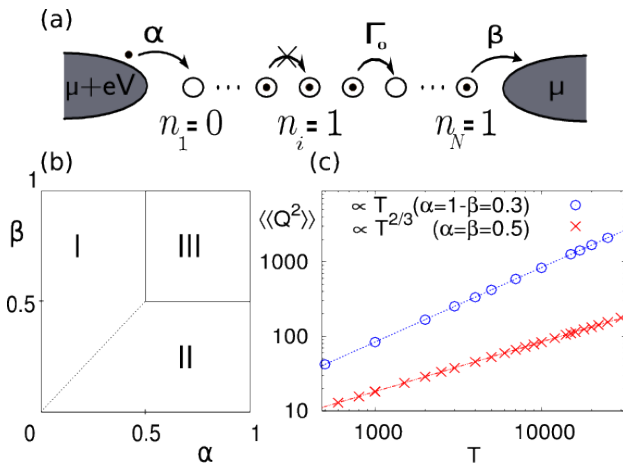
- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.
- Non-Hermitian Spin Chain: **ASEP is Integrable model solvable by Bethe Ansatz**

APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

Current in a quantum-dot chain

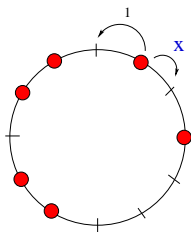
T. Karzig, F. von Oppen, *Signature of critical full counting statistics in a quantum-dot chain*, PRB **81**, 045317 (2010).



1. Large deviations of the current in a closed ring
(S. Prolhac)
2. Fluctuations of the current in an open system
(A. Lazarescu)
3. Corner Dynamics in 2 and 3 dimensions

1. Current Fluctuations on a ring

Markov Equation for the ASEP on a ring



L SITES
N PARTICLES

$\Omega = \binom{L}{N}$
CONFIGURATIONS

x asymmetry parameter

Master Equation for the Probability $P_t(x_1, \dots, x_N)$ of being in configuration $1 \leq x_1 < \dots < x_N \leq L$ at time t .

$$\begin{aligned} \frac{dP_t}{dt} &= \sum_i' [P_t(x_1, \dots, x_i - 1, \dots, x_N) - P_t(x_1, \dots, x_i, \dots, x_N)] \\ &+ x \sum_i' [P_t(x_1, \dots, x_i + 1, \dots, x_N) - P_t(x_1, \dots, x_i, \dots, x_N)] \\ &= MP. \end{aligned}$$

The sum being restricted to admissible configurations.

ASEP: An Integrable System

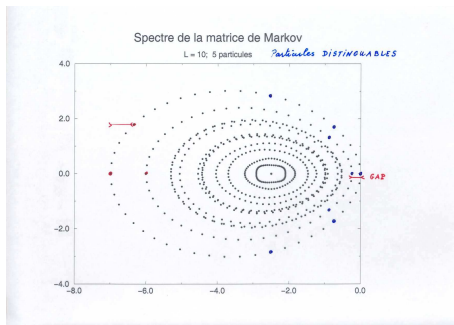
MAPPING TO A NON-HERMITIAN SPIN CHAIN

$$M = \sum_{l=1}^L \left(\mathbf{s}_l^+ \mathbf{s}_{l+1}^- + x \mathbf{s}_l^- \mathbf{s}_{l+1}^+ + \frac{1+x}{4} \mathbf{s}_l^z \mathbf{s}_{l+1}^z - \frac{1+x}{4} \right)$$

Complex Eigenvalues $M\psi = E\psi$:

- **Ground State** : $E = 0$, $P = \Omega^{-1}$ (non-degenerate).
- **Excited States** : $\Re(E) < 0$ (Perron-Frobenius).

Excitations correspond to relaxation times.



Integrability of ASEP: Bethe Ansatz

Eigenvector ψ of M written as a linear combination of plane waves, with pseudo-momenta given by z_1, \dots, z_N :

$$\psi(x_1, \dots, x_N) = \sum_{\sigma \in \Sigma_N} \mathcal{A}_\sigma \prod_{i=1}^N z_{\sigma(i)}^{x_i}$$

The **Bethe Equations** provide us with the quantification of the z_i 's:

$$z_i^L = (-1)^{N-1} \prod_{j=1}^N \frac{x z_i z_j - (1+x) z_i + 1}{x z_i z_j - (1+x) z_j + 1}$$

The corresponding eigenvalue is given by

$$E(z_1, z_2 \dots z_N) = \sum_{i=1}^N \frac{1}{z_i} + x \sum_{i=1}^N z_i - N(1+x).$$

The special case of TASEP

$$\text{TASEP : } x = 0$$

Eigenvectors of M as linear combinations of plane waves, with pseudo-momenta given by z_1, \dots, z_N :

$$\psi(x_1, \dots, x_N) = \det \left(\frac{2^{x_j} (z_i + 1)^{j-x_j}}{(z_i - 1)^j} \right) \quad \text{for } 1 \leq i, j \leq N$$

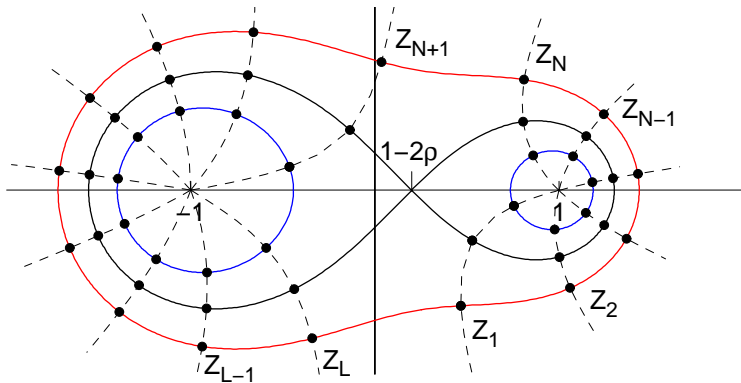
- ψ is an **eigenfunction** with **eigenvalue** $\mathbf{E} = \frac{1}{2}(-\mathbf{N} + \sum_j z_j)$.
- Cancellation of the two-particle collision terms ($x_{k-1} = x_k - 1$).
- **Bethe Equations**

$$(1 - z_i)^N (1 + z_i)^{L-N} = -2^L \prod_{j=1}^N \frac{z_j - 1}{z_j + 1} \quad \text{for } i = 1, \dots, N$$

Note that the r.h.s. is a constant independent of i : DECOUPLING.

Labelling the roots of the Bethe Equations

The loci of the roots (for $x = 0$) are remarkable curves (O. Golinelli):
The Cassini Ovals



Calculation of the GAP

The first excited state is solution of a transcendental equation. For a density ρ :

$$E_1 = -2\sqrt{\rho(1-\rho)} \frac{6.509189337\dots}{L^{3/2}} \pm \frac{2i\pi(2\rho-1)}{L}.$$

RELAXATION OSCILLATIONS

- Non-diffusive: Largest relaxation time $T \sim L^z$ with $z = 3/2$ (*D. Dhar, L.H. Gwa and H. Spohn, D. Kim*).
- Oscillations \rightarrow Traveling waves probed by dynamical correlations (*M. Barma, S. Majumdar, P. Krapivsky*).
- Classification of higher excitations (*J. de Gier and F.H.L. Essler, 2006*).

Large Deviations of the Current

Let Y_t be the **total current** i.e. total distance covered by all the N particles, hopping on a ring of size L , between time 0 and time t .

In the stationary state, a non-vanishing mean-current: $\frac{Y_t}{t} \rightarrow J$

The fluctuations of Y_t obey a **Large Deviation Principle**:

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

$\Phi(j)$ being the **large deviation function** of the total current.

Equivalently, consider the **moment-generating function**, which when $t \rightarrow \infty$, behaves as

$$\langle e^{\mu Y_t} \rangle \simeq e^{E(\mu)t}$$

The statistics of Y_t is encoded by $\Phi(j)$ or $E(\mu)$, that are related by a **Legendre transform**

$$E(\mu) = \max_j (\mu j - \Phi(j))$$

Bethe Ansatz for current statistics

This function $E(\gamma)$ is the maximal eigenvalue of a deformation of the Markov Matrix $M(\gamma) = M_0 + e^\gamma M_+ + e^{-\gamma} M_-$

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz.

There exist two polynomials $Q(T)$ and $R(T)$ such that

$$Q(T)R(T) = e^{L\gamma}(1-T)^L Q(xT) + x^N(1-xT)^L Q(T/x)$$

where $Q(T)$ of degree N vanishes at the Bethe roots. (Functional Bethe Ansatz): a purely algebraic problem, solved perturbatively w.r.t. γ .

Knowing $Q(T)$, we obtain an expansion of $E(\gamma)$. This provides the full statistics of the current and its large deviations.

(The $x = 0$ case was solved by B. Derrida and J. L. Lebowitz, 1998).

Cumulants of the Current

- Mean Current: $J = (1 - x) \frac{N(L-N)}{L-1}$

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- Third cumulant (Skewness):

$$\begin{aligned}
 \frac{E_3}{6L^2} &= \frac{1-x}{L-1} \sum_{i>0} \sum_{j>0} \frac{C_L^{N+i} C_L^{N-i} C_L^{N+j} C_L^{N-j}}{(C_L^N)^4} (i^2 + j^2) \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} \\
 &- \frac{1-x}{L-1} \sum_{i>0} \sum_{j>0} \frac{C_L^{N+i} C_L^{N+j} C_L^{N-i-j}}{(C_L^N)^3} \frac{i^2 + ij + j^2}{2} \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} \\
 &- \frac{1-x}{L-1} \sum_{i>0} \sum_{j>0} \frac{C_L^{N-i} C_L^{N-j} C_L^{N+i+j}}{(C_L^N)^3} \frac{i^2 + ij + j^2}{2} \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} \\
 &- \frac{1-x}{L-1} \sum_{i>0} \frac{C_L^{N+i} C_L^{N-i}}{(C_L^N)^2} \frac{i^2}{2} \left(\frac{1+x^i}{1-x^i} \right)^2 \\
 &+ \frac{1-x}{L-1} \left[\frac{N(L-N)}{4(2L-1)} \frac{C_{2L}^{2N}}{(C_L^N)^2} - \frac{N(L-N)}{6(3L-1)} \frac{C_{3L}^{3N}}{(C_L^N)^3} \right]
 \end{aligned}$$

Cumulants of the Current II

- Mean Current: $J \sim (1-x)L\rho(1-\rho)$ for $L \rightarrow \infty$
- Diffusion Constant:

$$D \sim 4\phi L\rho(1-\rho) \int_0^\infty du \frac{u^2}{\tanh \phi u} e^{-u^2}$$

when $L \rightarrow \infty$ and $x \rightarrow 1$ with fixed value of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

- Third cumulant (Skewness): \rightarrow Non Gaussian fluctuations.

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}} \right) \pi (\rho(1-\rho))^2 L^3$$

From the **Physics** point of view, the solution allows one to

- Classify the different **universality** classes (KPZ, EW).
- Study the various **scaling** regimes.
- Investigate the **hydrodynamic** behaviour.

The weakly symmetric case

For large system sizes, $L \rightarrow \infty$, in the scaling limit $x = 1 - \frac{\nu}{L}$, the cumulant generating function is given by

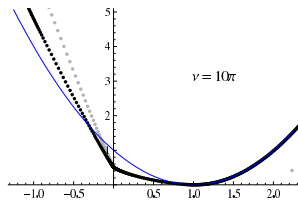
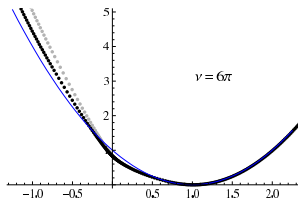
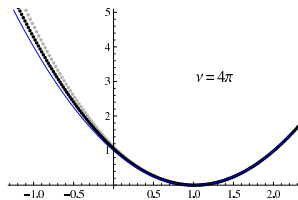
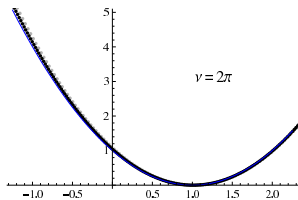
$$E\left(\frac{\mu}{L}\right) \simeq \frac{\rho(1-\rho)(\mu^2 + \mu\nu)}{L} - \frac{\rho(1-\rho)\mu^2\nu}{2L^2} + \frac{1}{L^2}\psi[\rho(1-\rho)(\mu^2 + \mu\nu)]$$

$$\text{with } \psi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$$

- The B_j 's are **Bernoulli Numbers**.
- Leading order (in $1/L$): **Gaussian** fluctuations.
- Subleading (in $1/L^2$): **Non-Gaussian** correction.
- **Phase transition** (predicted by *T. Bodineau* and *B. Derrida*) when

$$\nu \geq \nu_c = \frac{2\pi}{\sqrt{\rho(1-\rho)}}$$

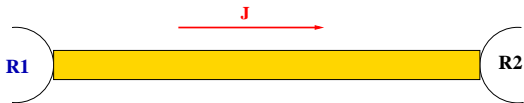
Behaviour of the large deviation function



2. Current Fluctuations in the open ASEP

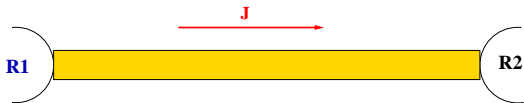
The Current in the Open System

The fundamental paradigm

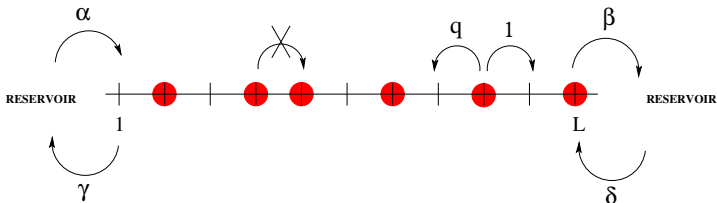


The Current in the Open System

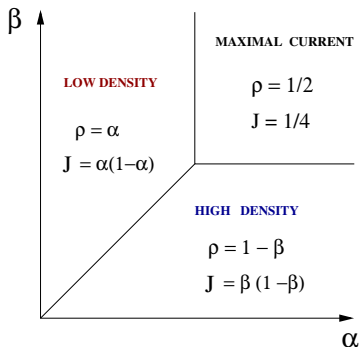
The fundamental paradigm



The asymmetric exclusion model with open boundaries



Phase Diagram (TASEP $q=0$)



This exact result is obtained using The Matrix Ansatz (DEHP, 1993). Stationary probabilities are written as traces over a suitable quadratic algebra.

Large Deviations of the Current: Framework

Let N_t be the total (time-integrated) number of particles exchanged between the system and the left reservoir between 0 and t . When a particle **enters** or **leaves** the system at site 1:

$$N_t = N_t \pm 1$$

- **Expectation value:** $\lim_{t \rightarrow \infty} \frac{\langle N_t \rangle}{t} = J(q; \alpha, \beta, \gamma, \delta, L)$
- **Variance:** $\lim_{t \rightarrow \infty} \frac{\langle N_t^2 \rangle - \langle N_t \rangle^2}{t} = \Delta(q; \alpha, \beta, \gamma, \delta, L)$
- **Cumulant Generating Function:** $\langle \exp(\mu N_t) \rangle \simeq \exp(E(\mu)t)$

$$E(\mu) = J\mu + \Delta \frac{\mu^2}{2} + E_3 \frac{\mu^3}{3!} + \dots$$

The **Large-Deviation Function** $\Phi(j)$ of the total current

$$P\left(\frac{N_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

is the *Legendre transform* of the Cumulant Generating Function $E(\mu)$.

A 'simple' case

In the case $q = 0$ and $\alpha = \beta = 1$ a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = - \sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]! [k(L+2)]!} \frac{B^k}{2k},$$

$$E = - \sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1) - 2]!}{[k(L+1) - 1]! [k(L+2) - 1]!} \frac{B^k}{2k}.$$

First cumulants of the current

- **Mean Value** : $J = \frac{L+2}{2(2L+1)}$

- **Variance** : $\Delta = \frac{3}{2} \frac{(4L+1)! [L!(L+2)]^2}{[(2L+1)!]^3 (2L+3)!}$

- **Skewness** :

$$E_3 = 12 \frac{[(L+1)!]^2 [(L+2)!]^4}{(2L+1)! [(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)! [(2L+2)!]^2 [(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$$

For large systems: $E_3 \rightarrow \frac{2187-1280\sqrt{3}}{10368} \pi \sim -0.0090978\dots$

Full Current Statistics of TASEP

For $q = 0$ and arbitrary (α, β) , the parametric representation of $E(\mu)$ is

$$\begin{aligned}\mu &= -\sum_{k=1}^{\infty} C_k(\alpha, \beta) \frac{B^k}{2k} \\ E &= -\sum_{k=1}^{\infty} D_k(\alpha, \beta) \frac{B^k}{2k}\end{aligned}$$

with

$$C_k(\alpha, \beta) = \oint_{\{0, a, b\}} \frac{dz}{2i\pi} \frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha, \beta) = \oint_{\{0, a, b\}} \frac{dz}{2i\pi} \frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}, \quad a = \frac{1-\alpha}{\alpha}, \quad b = \frac{1-\beta}{\beta}$$

Some explicit expressions

- **Mean Current:** (Same expression as in DEHP)

$$J = \frac{D_1(\alpha, \beta)}{C_1(\alpha, \beta)}$$

- **Fluctuations:**

$$\Delta = \frac{D_1 C_2 - D_2 C_1}{C_1^3}$$

- **Saddle point analysis in the low density phase:** ($\rho = \alpha$)

$$E_1 = \rho(1 - \rho)$$

$$E_2 = \rho(1 - \rho)(1 - 2\rho)$$

$$E_3 = \rho(1 - \rho)(1 - 6\rho + 6\rho^2)$$

$$E_4 = \rho(1 - \rho)(1 - 2\rho)(1 - 12\rho + 12\rho^2)$$

$$E_5 = \rho(1 - \rho)(1 - 30\rho + 150\rho^2 - 240\rho^3 + 120\rho^4) \dots$$

Asymptotics in the TASEP Phase Diagram

In the limit $L \rightarrow \infty$ of systems of large size, we have

- **Maximal Current phase** $\alpha > 1/2$ and $\beta > 1/2$: Cumulants are independent from α and β

$$E_k \sim \pi(\pi L)^{k/2-3/2} \text{ for } k \geq 2$$

- **Low Density phase** $\alpha < \min(\beta, 1/2)$:

$$E(\mu) = \frac{a}{a+1} \frac{e^\mu - 1}{e^\mu + a}$$

By Legendre Transform, the current Large Deviation Function is

$$\Phi(j) = \alpha - r + r(1-r) \ln \left(\frac{1-\alpha}{\alpha} \frac{r}{1-r} \right)$$

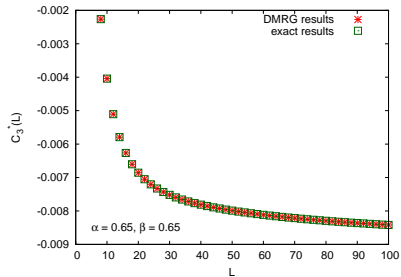
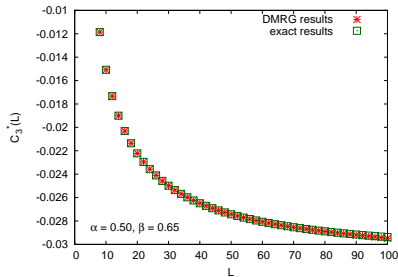
where the current j is parametrized as $j = r(1-r)$.

Agrees with Macroscopic Fluctuation Theory (T. Bodineau and B. Derrida).

- **Along the shock line** $\alpha = \beta \leq 1/2$, fluctuations are enhanced

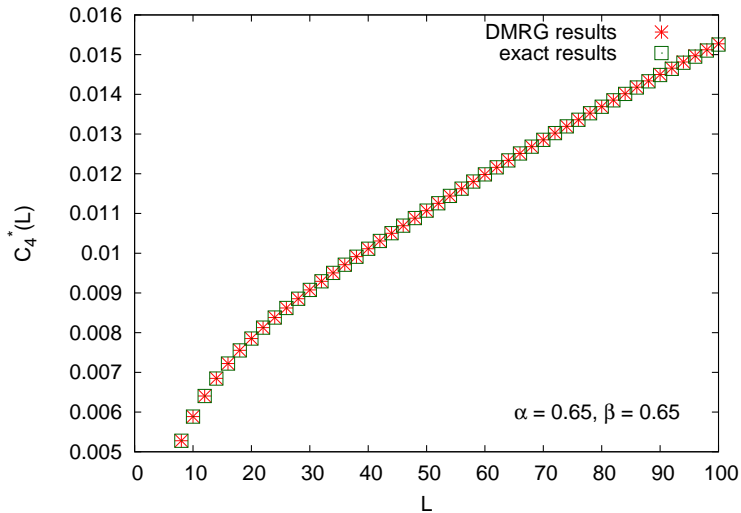
$$E_k \simeq \epsilon_k \alpha (1-\alpha)(1-2\alpha)^{k-1} L^{k-2} \text{ for } k \geq 2$$

DMRG Results (M. Gorissen, C. Vanderzande)

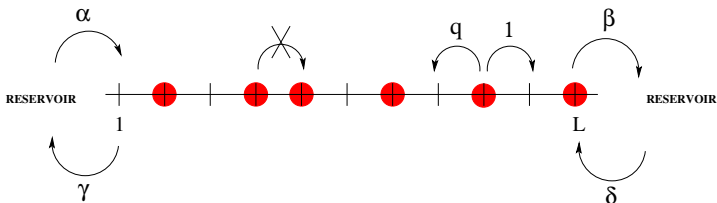


SKEWNESS

Fourth Cumulant (DMRG)



Current fluctuations in the general ASEP



For arbitrary values of q and $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\mu = - \sum_{k=1}^{\infty} C_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k} \quad \text{and} \quad E = - \sum_{k=1}^{\infty} D_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

The coefficients C_k and D_k are given by contour integrals in the complex plane:

$$C_k = \oint_C \frac{dz}{2i\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_C \frac{dz}{2i\pi} \frac{\phi_k(z)}{(z+1)^2}$$

Structure of the solution

The auxiliary function $W_B(z) = \sum_{k \geq 1} \phi_k(z) \frac{B^k}{k}$ solves a functional Bethe equation:

$$W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

- The operator X is an integral operator

$$X[W_B](z_1) = \oint_C \frac{dz_2}{i2\pi z_2} W_B(z_2) K(z_1, z_2)$$

$$\text{with kernel } K(z_1, z_2) = 2 \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}$$

- The function $F(z)$ is given by

$$F(z) = \frac{(1+z)^\infty (1+z^{-1})^\infty (z^2)^\infty (z^{-2})^\infty}{(a+z)^\infty (a+z^{-1})^\infty (a-z)^\infty (a-z^{-1})^\infty (b+z)^\infty (b+z^{-1})^\infty (b-z)^\infty (b-z^{-1})^\infty}$$

where $(x)^\infty = \prod_{k=0}^{\infty} (1 - q^k x)$ and a_\pm, b_\pm are simple functions of the boundary rates.

Une longue quête...

Variance of the TASEP Current (B. Derrida, M. R. Evans, K. M., 1995).

An old Formula for Δ :

$$\begin{aligned} \Delta = & \frac{\langle \alpha | C^{N-1} | \beta \rangle}{\langle \alpha | C^N | \beta \rangle} - \frac{\langle \alpha | C^{N-1} | \beta \rangle}{\langle \alpha | C^N | \beta \rangle} \sum_{n=1}^{N-1} \left[\frac{\langle \alpha | C^{n-1} D C^{N-n} | \beta \rangle}{\langle \alpha | C^n | \beta \rangle} \right. \\ & \left. - \frac{\langle \alpha | C^{n-1} D C^{N-1-n} | \beta \rangle}{\langle \alpha | C^{n-1} | \beta \rangle} \right] \\ & + \sum_{n=0}^{N-1} \frac{(2n)!}{n!(n+1)!} \left[(N-n-1) \frac{\langle \alpha | C^{N-n-2} | \beta \rangle}{\langle \alpha | C^n | \beta \rangle} \right. \\ & \left. - (N-n+1) \frac{\langle \alpha | C^{N-1} | \beta \rangle}{\langle \alpha | C^n | \beta \rangle} \frac{\langle \alpha | C^{N-n-1} | \beta \rangle}{\langle \alpha | C^n | \beta \rangle} \right] \\ & + 2 \frac{\langle \alpha | C^{N-1} | \beta \rangle}{\langle \alpha | C^N | \beta \rangle^2} \sum_{n=1}^N \langle \alpha | C^{N-n} | \beta \rangle \\ & \times [\langle \alpha | C^{n-1} D C^n | \beta \rangle - \langle \alpha | C^n D C^{n-1} | \beta \rangle] \\ & - \frac{1}{\langle \alpha | C^N | \beta \rangle^2} \sum_{n=1}^{N-1} \langle \alpha | C^{N-n} | \beta \rangle \\ & \times [\langle \alpha | C^{n-1} D C^{N-1} | \beta \rangle - \langle \alpha | C^{N-1} D C^{n-1} | \beta \rangle] \\ & - \frac{1}{\langle \alpha | C^N | \beta \rangle^2} \sum_{n=1}^{N-1} \langle \alpha | C^{N-n-1} | \beta \rangle \\ & \times [\langle \alpha | C^{n-1} D C^n | \beta \rangle - \langle \alpha | C^{N-1} D C^n | \beta \rangle] \end{aligned}$$

where the matrix elements involved are given by expressions derived in ref. 14:

$$\langle \alpha | C^n | \beta \rangle = \frac{\alpha \beta}{\alpha - \beta} \left[R_N \left(\frac{1}{\beta} \right) - R_N \left(\frac{1}{\alpha} \right) \right] \langle \alpha | \beta \rangle \quad (59)$$

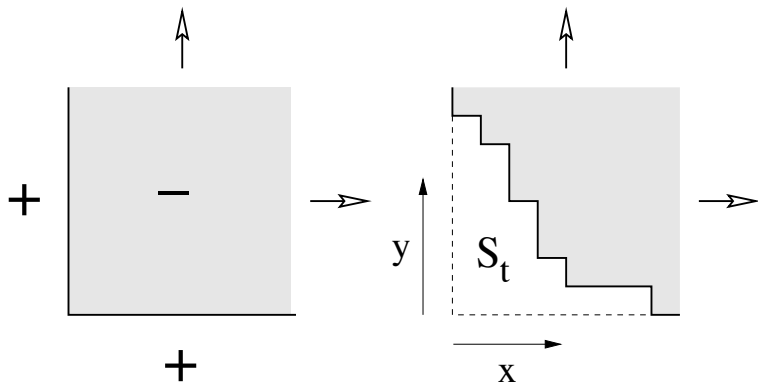
$$\langle \alpha | C^m D C^n | \beta \rangle = \sum_{p=0}^{n-1} \frac{(2p)!}{p!(p+1)!} \langle \alpha | C^{m+n-p} | \beta \rangle + R_n \left(\frac{1}{\beta} \right) \langle \alpha | C^m | \beta \rangle \quad (60)$$

where

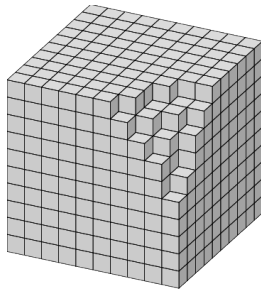
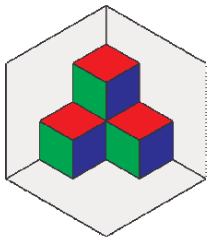
$$\begin{aligned} R_n \left(\frac{1}{\beta} \right) &= \sum_{p=1}^n \frac{p(2n-1-p)!}{n!(n-p)!} \frac{1}{\beta^{p+1}} \quad \text{for } n \geq 1 \\ R_0 \left(\frac{1}{\beta} \right) &= \frac{1}{\beta} \end{aligned} \quad (61)$$

3. Shapes of Growth

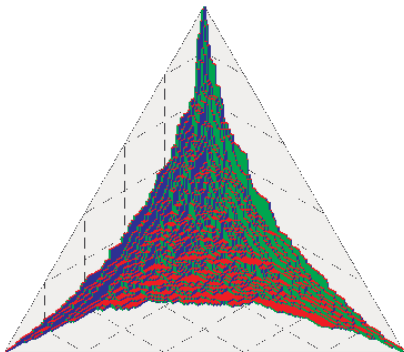
A crystal growing on a corner in two dimensions



Corner Growth/Melting in three dimensions



Corner Growth in three dimensions



The Wulff Construction

What is the shape of a large crystal at thermodynamic equilibrium?

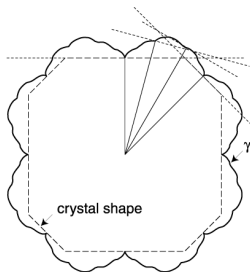
One has to minimize the surface free energy for a fixed total volume

The surface free energy σ depends on the **local orientation** of the crystal, given by the normal direction \vec{n} . The **Wulff diagram** is the surface (or curve)

$$\vec{n} \rightarrow \sigma(\vec{n})\vec{n}$$

The solution of the variational problem is purely geometrical:

La forme du cristal est la podaire (pedal) de la surface de Wulff



Application to the 2d Ising Model at equilibrium

$$E = -J \sum_{\langle ij \rangle} S_i S_j \quad \text{with} \quad 0 \leq T \leq T_c$$

What is the shape of a large domain of up spins immersed in a sea of down spins?

Duality Relation between surface tension and correlation length:

$$\sigma(\theta) = \frac{kT}{\xi^*(\theta)}$$

The correlation length is given by

$$\xi^*(\theta) = - \lim_{L \rightarrow \infty} \frac{1}{L} \ln \langle S_{00} S_{MN} \rangle \quad \text{with} \quad (M, N) = (L \cos \theta, L \sin \theta)$$

An explicit formula for this quantity is known (Mc Coy and Wu, '70's) and so is $\sigma(\theta)$: one can draw the Wulff crystal for the Ising model.

C. Rottman, M. Wortis, PRB **24**, 6274 (1981).

Y. Avron, H. van Beijeren, L.S. Schulman, R.K.P. Zia, J. Phys. A **15**, L81 (1982).

Limiting shapes of Young Tableaux

A Young Tableau of total size N encodes **partitions** of the integer N :

$$N = r_1 + r_2 + \dots + r_p \quad \text{with} \quad r_1 \geq r_2 \geq \dots \geq r_p$$

An equivalent representation is to use 'bosonic' occupation variables:

$$N = \sum_{k=0}^{\infty} kn_k$$

The mapping from one representation to the other is obtained via

$$r_l = \sum_{k=l}^{\infty} n_k$$

Let us study the partitions of N with **uniform measure**:

$$\text{Proba}_N(n_1, n_2, \dots, n_k, \dots) = \frac{1}{p(N)} \delta\left(\sum_{k=0}^{\infty} kn_k - N\right)$$

where $p(N) = \sum_{\{n_k\}} \delta(\sum_{k=0}^{\infty} kn_k - N)$ is the **partition number of N** .

Some properties of the partition number

The generating function of the partition numbers was found by Euler:

$$\sum_N p(N) x^N = \sum_{\{n_k\}} x^{\sum_k k n_k} = \prod_k \frac{1}{1 - x^k}$$

Inversely, the partition number can be extracted as follows

$$p(N) = \frac{1}{2i\pi} \oint \frac{dz}{z^{N+1}} \exp\left(-\sum_k \ln(1 - z^k)\right)$$

For large N , the contribution of the dominant singularity $z = 1$. The saddle-point method yields the celebrated result by Hardy and Ramanujan:

$$p(N) \sim \exp\left(\pi\sqrt{2N/3}\right)$$

The result of Vershik (1996)

The same technique can be adapted to calculate the asymptotic shape of a Young Tableau of size N :

$$\ell \rightarrow \langle r_\ell \rangle_N = \sum_{k \geq \ell} \langle n_k \rangle_N$$

Rescaling the height function as $\ell = x \frac{\sqrt{6N}}{\pi}$ and $r = y \frac{\sqrt{6N}}{\pi}$, the limiting shape of Young Tableaux, drawn with uniform probability, is given by (Vershik, 1996):

$$\exp(-x) + \exp(-y) = 1$$

Borodin, Okounkov and Olshanki (2000) studied the fluctuations of the shape and proved that it is a determinantal process with a sine kernel.

This limit shape can also be obtained by a variational principle *à la Wulff* by noting that the frontier of a tableau is South-East directed walk and by enumerating such walks.

Effects of the statistics

Another natural measure on Young Tableaux is the Plancherel measure

$$\mu(\lambda) = \frac{(f^\lambda)^2}{N!}$$

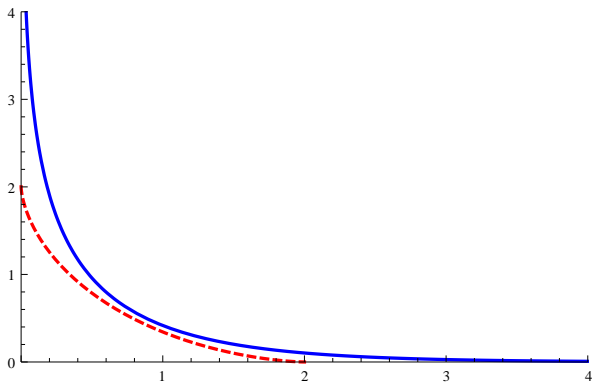
The limit shape of Young Tableaux, drawn with the Plancherel measure, was found by Logan and Shepp and by Vershik and Kerov (1977)

$$\begin{aligned}x &= y + 2 \cos \theta \\y &= \frac{2}{\pi} (\sin \theta - \theta \cos \theta)\end{aligned}$$

where the axis-coordinates have been scaled by \sqrt{N} and $0 \leq \theta \leq \pi$.

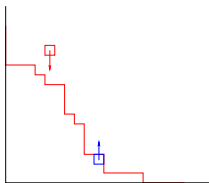
Note that $x_{max} = 2\sqrt{N}$: this is precisely the mean-length of the longest increasing subsequence extracted from N randomly ordered numbers (Ulam's problem, restated by Schensted).

Comparison: Plancherel versus uniform measure



Far from equilibrium

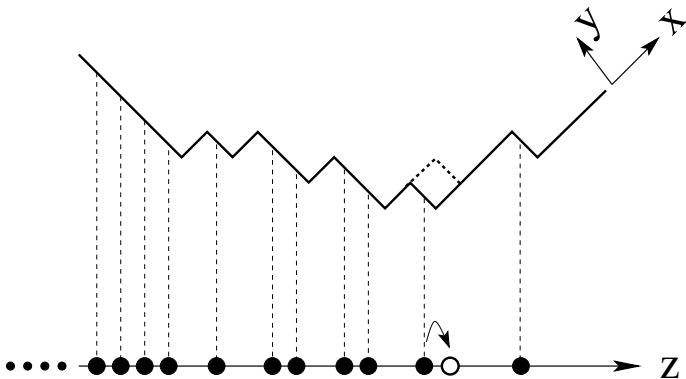
We now focus on the non-equilibrium growth of a Young diagram:
Evolution of a corner under Glauber dynamics at a vanishing low temperature.



Evaporation/Deposition events that respect the monotonicity constraint (no over-hangs). **Different types of dynamics:**

- (i) Deposition and Evaporation with the **same rate** (Zero magnetic field).
- (ii) Deposition with rate 1. **No evaporation** (Non-Zero magnetic field).
- (iii) Deposition and Evaporation with **different rates** (Vanishingly small field). A negative field stabilizes the phase inside the corner and leads to a finite size equilibrium shape.

Mapping to a one-dimensional particle process



Evolution of the mean shape

Thanks to this mapping the shape of the crystal corresponds to the density profile in the particle language.

In the case of deposition only, H. Rost (1981) found the limiting shape of the crystal at time t from the TASEP:

$$\sqrt{x} + \sqrt{y} = \sqrt{t}$$

For evaporation and deposition occurring at the same rate (zero magnetic field), the limiting shape was calculated only recently:

$$\eta = \frac{1}{\sqrt{4\pi}} e^{-(\xi-\eta)^2} - \frac{\xi-\eta}{\sqrt{\pi}} \int_{\xi-\eta}^{\infty} d\zeta e^{-\zeta^2}$$

where $\xi = \frac{x}{\sqrt{4t}}$, and $\eta = \frac{y}{\sqrt{4t}}$. In particular, the diagonal $x = y$ crosses the interface at $\xi = \eta = (4\pi)^{-1/2}$ and therefore $x = y = \sqrt{t/\pi}$.

Statistics of the apex height

The height above the origin (i.e. the intersection of the diagonal $x = y$ with the interface) corresponds to the total current Q_t that has flown through the $(0,1)$ bond in the exclusion process.

The statistics of Q_t was investigated by K. Johansson in the TASEP case. They found

$$Q_t = \frac{t}{4} + \frac{t^{1/3}}{2^{4/3}} \chi$$

where χ is a random variable distributed according to

$$\text{Prob}(\chi \leq s) = 1 - F_2(-s)$$

$$\text{with } F_2(s) = \exp\left(-\int_s^\infty (x-s) u(x)^2 dx\right)$$

$u(x)$ being the solution of Painlevé II equation $u'' = xu + 2u^3$, matching the Airy function at infinity. The Tracy-Widom function F_2 is the cumulative distribution of the maximal eigenvalue λ_{max} in a GUE.

The SEP case was studied recently by Derrida and Gerschenfeld.

Statistics of the Area

The area of the molten region can be written in terms of the displacements of the particles

$$S_t = \sum_{\text{all particles}} \text{displacements}$$

It is more convenient to use the representation of the area in terms of local Boolean variables $\tau_x(t) = 0, 1$ which indicate if site x is empty or occupied at time t :

$$S_t = \sum_{x=-\infty}^{+\infty} x[\tau_x(t) - \tau_x(0)]$$

We want to compute the statistics of S_t .

Some results

At all times, we have

$$\langle S_t \rangle = t$$

The variance of the area is given by

$$\lim_{t \rightarrow \infty} \frac{\langle S_t^2 \rangle - \langle S_t \rangle^2}{t^{3/2}} = \frac{4\sqrt{2}}{3\sqrt{\pi}} = 1.063846080\dots$$

The probability that the quadrant is in the initial state at a large time t , i.e. the probability that $S_t = 0$:

$$\lim_{t \rightarrow \infty} \frac{\ln P_0(t)}{\sqrt{t}} = -\frac{1}{\sqrt{\pi}} \zeta\left(\frac{3}{2}\right) = -1.473874960\dots$$

The large deviation property of the surface is given by

$$\text{Prob}\left(\frac{S_t}{t} = s\right) \sim e^{-\sqrt{t}\phi(s)}$$

where $\phi(s) \sim (s-1)^2$ in the vicinity of $s = 1$.

Relation with Bethe Ansatz (Tracy and Widom)

The Green function $P_t^{(n)}(y_1, y_2, \dots, y_n | x_1, x_2, \dots, x_n)$ is known from Bethe Ansatz and is given by

$$\sum_{\sigma} \epsilon(\sigma) \oint \left[\prod_{k=1}^n \frac{dz_k}{2i\pi z_k} e^{t(z_k+1/z_k-2)} z_{\sigma(k)}^{y_k-x_{\sigma(k)}} \right] \left[\prod_{k \langle l} \frac{z_{\sigma(k)} z_{\sigma(l)} + 1 - 2z_{\sigma(k)}}{z_k z_l + 1 - 2z_k} \right]$$

From this equation, one can calculate the values of the k-point correlations $\langle \tau_{y_1}(t) \dots \tau_{y_k}(t) \rangle$.

For the SEP, this allows us to calculate the moments of the area.

This is not true for the general case (totally or partially asymmetric): open problem.

3d Plane Partitions with Uniform Measure

In a plane partition, the height satisfies

$$0 \leq h(k+1, l) \leq h(k, l) \quad \text{and} \quad 0 \leq h(k, l+1) \leq h(k, l)$$

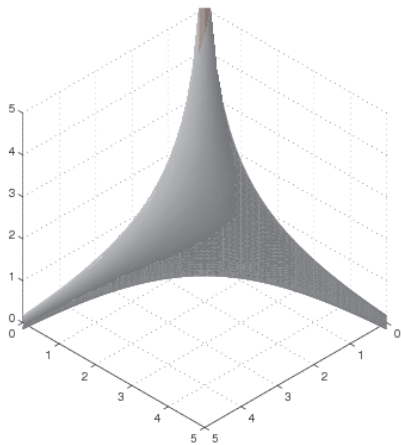
Consider plane partitions of total volume N , chosen randomly with uniform probability measure. The limiting shape of the corner crystal was found by [R. Cerf and R. Kenyon](#), and also by [A. Okounkov](#) (ca 2000).

The limiting surface is $\left(\frac{\zeta(3)}{4}\right)^{-1/3} S_0$ (after rescaling the coordinates by $N^{1/3}$), S_0 being given by the parametric equations

$$S_0 = \{(f(A, B, C) - \ln A, f(A, B, C) - \ln B, f(A, B, C) - \ln C)\}$$

A, B, C are strictly positive and

$$f(A, B, C) = \frac{1}{4\pi^2} \int_{[0, 2\pi]} \int_{[0, 2\pi]} \ln |A + Be^{i\theta} + Ce^{i\phi}| \, d\theta d\phi$$



Growing Plane Partitions

What is the limit shape for a Plane Partition growing with Glauber dynamics with non-zero magnetic field?

We conjecture the following governing equation for the interface profile:

$$z_t = \frac{z_x}{z_x - 1} \frac{z_y}{z_y - 1} \left[1 - \frac{1}{z_x + z_y} \right]$$

- We seeked an equation of the form $z_t = F(z_x, z_y)$ involving only first derivatives.
- The correct 3d equation has to reduce to the 2d equation of motion on the boundaries $x = 0$ or $y = 0$. In 2 dimensions, the dynamics of an interface $\zeta(\eta)$ obeys

$$\zeta_t = \frac{\zeta_\eta}{\zeta_\eta - 1}$$

- The equation $z_t = F(z_x, z_y)$ must be invariant under exchange of any pair of coordinates:

$$F\left(\frac{1}{a}, \frac{1}{b}\right) = -\frac{1}{a} F\left(a, -\frac{a}{b}\right)$$

The method of characteristics

Any first order PDE $F(x, y, z, z_x, z_y) = 0$ can be shown to be equivalent to the following set of first order ODE's:

$$x_t = F_q$$

$$y_t = F_r$$

$$z_t = qF_q + rF_r$$

$$q_t = -F_x - qF_z$$

$$r_t = -F_y - rF_z$$

where $q = z_x$ and $r = z_y$.

In the case we are interested in, the function F does not depend on the variable z . The equations for the characteristics become Hamiltonian.

Solution of the PDE

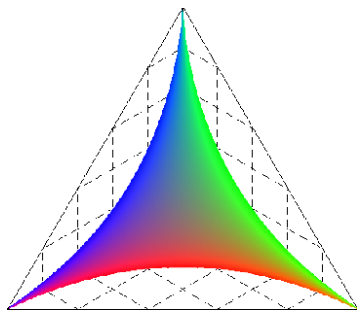
Using the method of characteristics, the conjectured equation can explicitly be solved:

$$\frac{x}{t} = A(q, r) \quad \frac{y}{t} = B(q, r) \quad \frac{z}{t} = C(q, r)$$

with

$$A = \frac{r^2}{(r-1)(q-1)(q+r)} \left[\frac{1}{q-1} + \frac{1}{q+r} \right]$$
$$B = \frac{q^2}{(r-1)(q-1)(q+r)} \left[\frac{1}{r-1} + \frac{1}{q+r} \right]$$
$$C = \frac{q^2 r^2}{(r-1)(q-1)(q+r)} \left[\frac{1}{q-1} + \frac{1}{r-1} \right]$$

Some numerical tests



This surface cuts the ground plane $z = 0$ along the curve $\sqrt{x} + \sqrt{y} = \sqrt{t}$. The intersection with the ray $x = y = z$ is predicted to be $\frac{t}{8}$. Numerical simulations give the speed 0.1261...

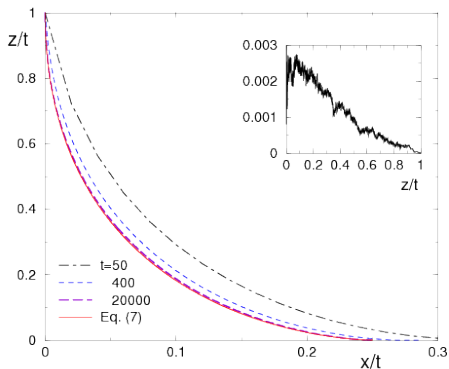
The volume can be calculated exactly and it grows as

$$V/t^3 = v = \frac{3\pi^2}{2^{11}} = 0.014457\dots$$

The numerical measurement gives $v \approx 0.01472(3)$, within 1.8% of the prediction.

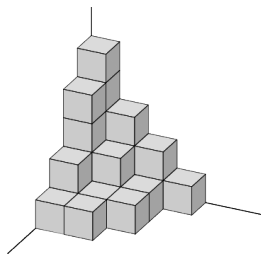
The intersection with the plane $x = y$ is given by

$$\frac{x}{t} = \frac{1}{2} \frac{z}{t} - \frac{3}{4} \left(\frac{z}{t} \right)^{2/3} + \frac{1}{4}$$

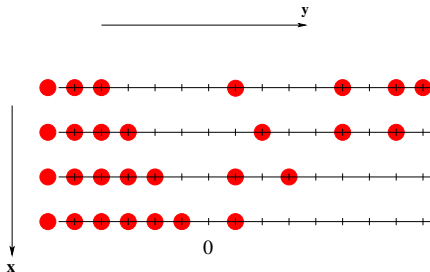


(Jason Olejarz, Sid Redner, P. K. and K. M: PRL **108** 016102 (2012))

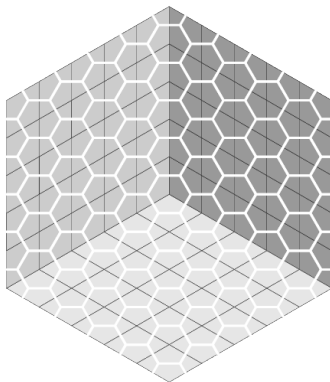
Equivalence with particle systems



Equivalent to family of coupled exclusion processes:



Dimers versus ABC model in the plane



More conjectures

- Diffusive Growth in 3d:

$$z_t = \frac{\left(1 - \frac{1}{z_x + z_y}\right)^2}{\left(1 - \frac{1}{z_x}\right)^2 \left(1 - \frac{1}{z_y}\right)^2} \left[\frac{z_{xx}}{z_x^2} + \frac{z_{yy}}{z_y^2} - \frac{z_{xy}}{z_x z_y} \right]$$

Higher Dimensions:

- Ballistic:

$$h_t = H = \prod_{1 \leq i_1 < \dots < i_p \leq d-1} \left(1 - \frac{1}{h_{i_1} + \dots + h_{i_p}}\right)^{(-1)^p}$$

- Diffusive:

$$\frac{h_t}{H^2} = \sum_{i=1}^{d-1} \frac{h_{ii}}{h_i^2} - \frac{2}{d-1} \sum_{i < j} \frac{h_{ij}}{h_i h_j}$$

A lot of work remains to be done to verify these conjectured equations and to extract from them other cogent tests.

Conclusions

Exact solutions of the asymmetric exclusion process are paradigms for the behaviour of systems far from equilibrium in low dimensions: Dynamical phase transitions, Non-Gibbsean measures, Large deviations, Fluctuations Theorems...

The large deviation functions (LDF) appear as the right generalization of the thermodynamic potentials: convex, optimized at the stationary state, and non-analytic features can be interpreted as phase transitions. Besides, the LDF's satisfy remarkable identities (Gallavotti-Cohen) valid far away from equilibrium. The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.

Interacting Particle Processes also represent Growth Models and can be used to investigate properties of dimer tilings, Young tableaux and plane partitions.