Alday-Gaiotto-Tachikawa conjecture and Integrability

based on papers with

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Pure N=2 SUSY (Short review)

$$L = 1/gTr(F^2 + i\theta F F_D + [\Phi, \Phi^*]^2 + |D\Phi|^2) + Fermions$$

We can add to this theory the matter supemultipletsmultiplets q which belong to some representation of gauge group:

$$\delta L = 1/gTr(|Dq|^2 + |\Phi q|^2 + |Mq|^2 + ...$$

Instantons are the solution to equation

$$F = F_D$$

The action on the instantons with topological number m is $m 8\pi^2/g^2$ Vacuum moduli space: $[\Phi, \Phi^*]^2 = 0$. For gauge group $SU(2)$ it can be represented as $\Phi = a\tau_3$. This destroys $SU(2)$ symmetry up to $U(1)$ and gives masses to vector particles (W-bosons). In the theory appear also magnetic charged particles (monopoles) which are the solution to equation

$$F_{ij} = \epsilon_{ijk} D_k \Phi$$

2D CFT (review)

- We have the complete set of local fields $\{\mathcal{O}_k(\xi)\}$

$$\mathcal{O}_i(\xi)\mathcal{O}_j(0) = \sum_k C_{ij}^k(\xi)\mathcal{O}_k(0).$$

- The structure constants $C_{ij}^k(\xi)$ are subject to associativity condition

- In CFT the set $\{\mathcal{O}_k(\xi)\}$ can be decomposed as

$$\{\mathcal{O}_k(\xi)\} = \sum_n [\Phi_n].$$

- The ancestor of each family Φ_n is called primary field

$$\Phi_n(z,\bar{z}) \longrightarrow \left(\frac{dw}{dz}\right)^{\Delta_n} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}_n} \Phi_n(w,\bar{w}), \quad z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z})$$

- Other representatives of $[\Phi_n]$ are called descendant fields

$$\Delta_n^{(k)} = \Delta_n + k, \qquad \bar{\Delta}_n^{(\bar{k})} = \bar{\Delta}_n + \bar{k},$$

- In two dimensions the conformal group is $\mathrm{Vir} \otimes \bar{\mathrm{V}}\mathrm{ir}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0},$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0},$$

- And hence the conformal family is a tensor product $[\Phi_n] = \pi_n \otimes \bar{\pi}_n$

$$[\Phi] = \{\Phi, \Phi^{(-1)}, \Phi^{(-1,-1)}, \Phi^{(-2)}, \ldots\} \otimes \{\ldots\}$$

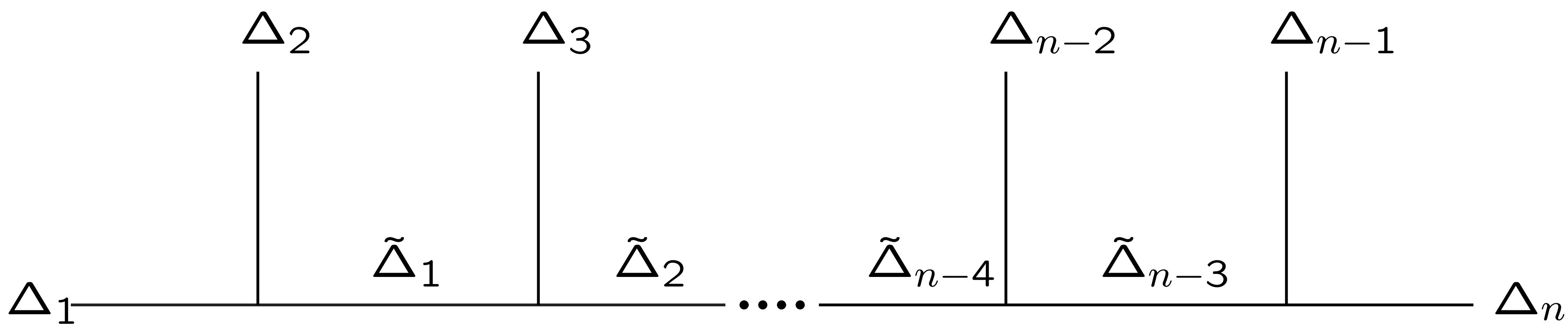
- One can show that OPE of primary fields has a form

$$\Phi_1\Phi_2 = \sum_k C_{12}^k \Big(\Phi_k + \beta_1 \Phi_k^{(-1)} + \beta_{1,1}\Phi_k^{(-1,-1)} + \beta_2 \Phi_k^{(-2)} + \ldots\Big) \otimes (\ldots)$$

- We can introduce the notion of the *conformal blocks*. They represent holomorphic contributions to the multi-point correlation function

$$\langle \Phi_1(z_1, \bar{z}_1) \dots \Phi_n(z_n, \bar{z}_n) \rangle$$

and can be represented as



- It is also convenient to choose $z_1 = 0$, $z_{n-1} = 1$, $z_n = \infty$ and

$$z_{i+1} = q_i q_{i+1} \dots q_{n-3} \quad \text{for} \quad 1 \le i \le n-3,$$

- Then the conformal block is a power series expansion

$$\mathcal{F}(q|\Delta_i, \tilde{\Delta}_j, c) = 1 + \sum_{\vec{k}} q_1^{k_1} q_2^{k_2} \dots q_{n-3}^{k_{n-3}} \mathfrak{F}_{\vec{k}}(\Delta_i, \tilde{\Delta}_j, c),$$

where the coefficients $\mathfrak{F}_{\vec{k}}(\Delta_i, \tilde{\Delta}_j, c)$ are some rational functions of Δ_i, $\tilde{\Delta}_j$ and the central charge c.

- There exists an algebraic procedure allowing to compute $\mathfrak{F}_{\vec{k}}(\Delta_i, \tilde{\Delta}_j, c)$ which is equivalent to the computation of the matrix elements

$$\langle i|L_{k'_1} \dots L_{k'_m} \Phi_k(1)\, L_{-k_n} \dots L_{-k_1}|j\rangle$$

using

$$\begin{aligned}
&L_n|j\rangle = 0 \;\; \langle j|L_{-n} = 0 \quad \text{for} \quad n > 0, \\
&L_0|j\rangle = \Delta_j|j\rangle \;\; \langle j|L_0 = \Delta_j\langle j| \\
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0}, \\
&[L_m, \Phi_k(z)] = \left(z^{m+1}\partial_z + (m+1)\Delta_k z^m\right)\Phi_k(z)
\end{aligned}$$

and

$$\langle i|\Phi_k(z)|j\rangle \sim z^{\Delta_i - \Delta_j - \Delta_k}$$

- Alday, Gaiotto and Tachikawa suggested to consider the function

$$Z(q|\Delta_i, \tilde{\Delta}_j, c) \stackrel{\text{def}}{=} \prod_{k=1}^{n-3} \prod_{m=k}^{n-3} (1 - q_k \dots q_m)^{2\alpha_{k+1}(Q-\alpha_{m+2})} \mathcal{F}(q|\Delta_i, \tilde{\Delta}_j, c),$$

where

$$\Delta_k = \alpha_k(Q - \alpha_k), \quad c = 1 + 6Q^2.$$

- They proposed that $Z(q|\Delta_i, \tilde{\Delta}_j, c)$ coincides with instanton part of the Nekrasov partition function for $\underbrace{U(2) \otimes \dots \otimes U(2)}_{n-3}$ $\mathcal{N} = 2$ supersymmetric gauge theory with 4 fundamental and $n-4$ bifundamental hypermultiplets and $q_m = \exp(8\pi^2/g_m^2 + i\theta_m)$

- The function $Z(q|\Delta_i, \tilde{\Delta}_j, c)$ has been computed by Nekrasov

$$Z(q|\Delta_i, \tilde{\Delta}_j, c) = 1 + \sum_{\vec{k}} q_1^{k_1} q_2^{k_2} \dots q_{n-3}^{k_{n-3}} Z_{\vec{k}}(\Delta_i, \tilde{\Delta}_j, c),$$

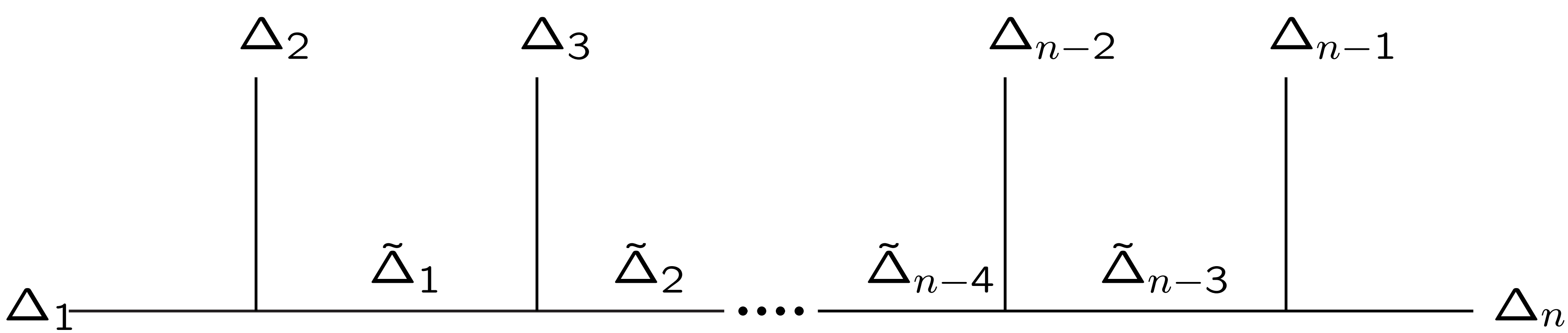
The coefficients $Z_{\vec{k}}(\Delta_i, \tilde{\Delta}_j, c)$ have explicit combinatorial expressions

$$Z_{\vec{k}}(\Delta_i, \tilde{\Delta}_j, c) = \sum_{\vec{\lambda}_1,\dots,\vec{\lambda}_{n-3}} Z_{\text{vec}}(P_1, \vec{\lambda}_1) \dots Z_{\text{vec}}(P_{n-3}, \vec{\lambda}_{n-3}) \times$$
$$\times Z_{\text{bif}}(\alpha_2|P, \varnothing; P_1, \vec{\lambda}_1) Z_{\text{bif}}(\alpha_3|P_1, \vec{\lambda}_1; P_2, \vec{\lambda}_2) Z_{\text{bif}}(\alpha_4|P_2, \vec{\lambda}_2; P_3, \vec{\lambda}_3) \times \dots$$
$$\dots \times Z_{\text{bif}}(\alpha_{n-2}|P_{n-4}, \vec{\lambda}_{n-4}; P_{n-3}, \vec{\lambda}_{n-3}) Z_{\text{bif}}(\alpha_{n-1}|P_{n-3}, \vec{\lambda}_{n-3}; \hat{P}, \varnothing).$$

where $\Delta_k = \alpha_k(Q - \alpha_k)$,

$$\Delta_1 = \frac{Q^2}{4} - P^2, \quad \Delta_n = \frac{Q^2}{4} - \hat{P}^2 \quad \text{and} \quad \tilde{\Delta}_j = \frac{Q^2}{4} - P_j^2,$$

and $\vec{\lambda} = (\lambda_1, \lambda_2)$ is the pair of Young diagrams such that $|\vec{\lambda}_j| = k_j$



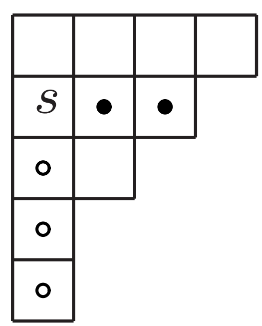
- The function Z_{bif} is given by $(Q = b + 1/b)$

$$Z_{\text{bif}}(\alpha|P', \vec{\mu}; P, \vec{\lambda}) = \prod_{i,j=1}^{2} \prod_{s\in\lambda_i} \Big(Q - E_{\lambda_i,\mu_j}\Big(P_i - P'_j\Big|s\Big) - \alpha\Big) \times$$
$$\times \prod_{t\in\mu_j} \Big(E_{\mu_j,\lambda_i}\Big(P'_j - P_i\Big|t\Big) - \alpha\Big),$$

where $\vec{P} = (P, -P)$, $\vec{P}' = (P', -P')$ and

$$E_{\lambda,\mu}\Big(P\Big|s\Big) = P - b\, l_{\mu}(s) + b^{-1}(a_{\lambda}(s) + 1).$$

- We choose the English convention to draw partitions. For example the partition $\lambda = (4, 3, 2, 1, 1)$ is drawn as follows



- $Z_{\text{vec}}(P, \vec{\lambda}) = 1/Z_{\text{bif}}(0|P, \vec{\lambda}; P, \vec{\lambda})$.

Special basis of states in Vir$\otimes\mathcal{H}$

- We consider the algebra $\mathcal{A} = \text{Vir} \otimes \mathcal{H}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0},$$
$$[a_n, a_m] = \frac{n}{2}\delta_{n+m,0}, \qquad [L_n, a_m] = 0.$$

- We will parametrize the central charge c of Virasoro algebra as

$$c = 1 + 6Q^2, \qquad \text{where} \quad Q = b + \frac{1}{b},$$

and define the primary field V_α as

$$V_\alpha \overset{\text{def}}{=} \mathcal{V}_\alpha \cdot V_\alpha^{\mathsf{L}},$$

where V_α^{L} is the primary field of Virasoro algebra and $\mathcal{V}_\alpha$:

$$\mathcal{V}_\alpha = e^{2(\alpha-Q)\varphi_-} e^{2\alpha\varphi_+},$$

with $\varphi_+(z) = i\sum_{n>0} \frac{a_n}{n} z^{-n}$ and $\varphi_-(z) = i\sum_{n<0} \frac{a_n}{n} z^{-n}$.

Proposition: There exists unique orthogonal basis $|P\rangle_{\vec{\lambda}}$ such that

$$\frac{{}_{\vec{\mu}}\langle P'|V_\alpha|P\rangle_{\vec{\lambda}}}{\langle P'|V_\alpha|P\rangle} = Z_{\mathsf{bif}}(\alpha|P',\vec{\mu};P,\vec{\lambda}).$$

We stress that the conjugation in the algebra $\mathcal{A}$ is defined as

$$\Big(L_{-k_n}\dots L_{-k_1}\Big)^+ = L_{k_1}\dots L_{k_n}, \qquad (a_{-n})^+ = a_n,$$

and the conjugation of the state $|P\rangle_{\vec{\lambda}}$ does not involve complex conjugation of its coefficients, i.e. for $|P\rangle_{\vec{\lambda}}$ given by

$$|P\rangle_{\vec{\lambda}} = \sum_{|\vec{\mu}|=|\vec{\lambda}|} C_{\vec{\lambda}}^{\mu_1,\mu_2}(P)\,\hat{a}_{-\mu_1}\hat{L}_{-\mu_2}|P\rangle,$$

we define conjugated state ${}_{\vec{\lambda}}\langle P|$ by

$${}_{\vec{\lambda}}\langle P| = \sum_{|\vec{\mu}|=|\vec{\lambda}|} C_{\vec{\lambda}}^{\mu_1,\mu_2}(P)\,\langle P|(\hat{a}_{-\mu_1})^+(\hat{L}_{-\mu_2})^+.$$

Examples:

$$|P\rangle_{\{1\},\varnothing} = -\left(L_{-1} + i(Q+2P)a_{-1}\right)|P\rangle,$$
$$|P\rangle_{\varnothing,\{1\}} = -\left(L_{-1} + i(Q-2P)a_{-1}\right)|P\rangle,$$

$$|P\rangle_{\{2\},\varnothing} = \Big(L_{-1}^2 - b^{-1}(Q+2P)L_{-2} + 2i(Q+b^{-1}+2P)L_{-1}a_{-1} - \\ -(Q+2P)(Q+b^{-1}+2P)a_{-1}^2 - ib^{-1}(Q+2P)(Q+b^{-1}+2P)a_{-2}\Big)|P\rangle,$$

$$|P\rangle_{\varnothing,\{2\}} = \Big(L_{-1}^2 - b^{-1}(Q-2P)L_{-2} + 2i(Q+b^{-1}-2P)L_{-1}a_{-1} - \\ -(Q-2P)(Q+b^{-1}-2P)a_{-1}^2 - ib^{-1}(Q-2P)(Q+b^{-1}-2P)a_{-2}\Big)|P\rangle,$$

$$|P\rangle_{\{1,1\},\varnothing} = \Big(L_{-1}^2 - b(Q+2P)L_{-2} + 2i(Q+b+2P)L_{-1}a_{-1} - \\ -(Q+2P)(Q+b+2P)a_{-1}^2 - ib(Q+2P)(Q+b+2P)a_{-2}\Big)|P\rangle,$$

$$|P\rangle_{\varnothing,\{1,1\}} = \Big(L_{-1}^2 - b(Q-2P)L_{-2} + 2i(Q+b-2P)L_{-1}a_{-1} - \\ -(Q-2P)(Q+b-2P)a_{-1}^2 - ib(Q-2P)(Q+b-2P)a_{-2}\Big)|P\rangle,$$

$$|P\rangle_{\{1\},\{1\}} = \Big(L_{-1}^2 - L_{-2} + 2iQL_{-1}a_{-1} + (1+4P^2-Q^2)a_{-1}^2 - iQa_{-2}\Big)|P\rangle.$$

Integrals of Motion

One can check that the states $|P\rangle_{\vec{\lambda}}$ are the eigenstates of the following infinite system of the Integrals of Motion

$$\mathbf{I}_2 = L_0 - \frac{c}{24} + 2\sum_{k=1}^{\infty} a_{-k}a_k,$$

$$\mathbf{I}_3 = \sum_{k=-\infty, k\neq 0}^{\infty} a_{-k}L_k + 2iQ\sum_{k=1}^{\infty} k a_{-k}a_k + \frac{1}{3}\sum_{i+j+k=0} a_i a_j a_k,$$

$$\mathbf{I}_4 = 2\sum_{k=1}^{\infty} L_{-k}L_k + L_0^2 - \frac{c+2}{12}L_0 + 6\sum_{k=-\infty, k\neq 0}^{\infty}\sum_{i+j=k} L_{-k}a_i a_j +$$

$$+ 12\left(L_0 - \frac{c}{24}\right)\sum_{k=1}^{\infty} a_{-k}a_k + 6iQ\sum_{k=-\infty, k\neq 0}^{\infty} |k|\, a_{-k}L_k +$$

$$+ 2(1-5Q^2)\sum_{k=1}^{\infty} k^2 a_{-k}a_k + 6iQ\sum_{i+j+k=0} |k|\, a_i a_j a_k + \sum_{i+j+k+l=0} : a_i a_j a_k a_l :$$

. . . .

Let us represent Virasoro generators L_n in terms of bosons c_k by

$$L_n = \sum_{k\neq 0,n} c_k c_{n-k} + i(nQ - 2\mathcal{P})c_n, \qquad L_0 = \frac{Q^2}{4} - \mathcal{P}^2 + 2\sum_{k>0} c_{-k}c_k,$$

$$[c_n, c_m] = \frac{n}{2}\delta_{n+m,0}, \qquad [\mathcal{P}, c_n] = 0, \qquad \mathcal{P}|P\rangle = P|P\rangle, \qquad \langle P|\mathcal{P} = -P\langle P|.$$

Proposition: The states $|P\rangle_{\lambda,\varnothing}$ and ${}_{\lambda,\varnothing}\langle P|$ can be defined as

$$|P\rangle_{\lambda,\varnothing} = \Omega_\lambda(P)\, \mathbf{J}_\lambda^{(1/g)}(x)|P\rangle, \qquad {}_{\lambda,\varnothing}\langle P| = \Omega_\lambda(P)\, \langle P|\, \mathbf{J}_\lambda^{(1/g)}(y),$$

where $g = -b^2$,

$$a_{-k} - c_{-k} = -ib\, p_k(x), \qquad a_k + c_k = -ib\, p_k(y),$$

with $p_k(x)$ being k-th power sum symmetric polynomial $p_k(x) = \sum_j x_j^k$ and $\mathbf{J}_\lambda^{(1/g)}(x)$ is the Jack polynomial associated with the Young diagram λ normalized as (“integral form” normalization)

$$\mathbf{J}_\lambda^{(1/g)}(x) = |\lambda|!\, m_{[1,\dots,1]}(x) + \dots,$$

where $m_{[\nu_1,\dots,\nu_n]}(x)$ is the monomial symmetric polynomial.

- The factor $\Omega_\lambda(P)$ is defined by

$$\Omega_\lambda(P) = (-b)^{|\lambda|} \prod_{(i,j)\in\lambda} (2P + ib + jb^{-1}),$$

index i runs vertically and j runs horizontally over the diagram λ.

- Calculation of matrix element $\frac{{}_{\mu,\varnothing}\langle P'|V_\alpha|P\rangle_{\lambda,\varnothing}}{\langle P'|V_\alpha|P\rangle}$.

There are infinite number of points α_n where the screening condition is satisfied:

$$P + P' + \alpha + nb = 0$$

In this case our matrix element possesses free field representation. We introduce screening charge:

$$\mathcal{S} = \int_C e^{2b\phi(\xi)} d\xi, \quad \phi = i\mathcal{P} \log \xi + i \sum_{k\neq 0} \frac{c_k}{k} \xi^{-k}$$

which commutes with Virasoro algebra, and introduce the screened vertex operator:

$$V^L_{\alpha_n}(z) = \mathcal{S}^n e^{2\alpha_n \phi(z)}$$

where the contours of integration start at point z and go around 0 counterclockwice.

The right and left operators A_{-k}, B_k

$$A_{-k} = a_{-k} - c_{-k}, \quad B_k = a_k + c_k$$

commute and have a simple CR with $e^{2b\phi(z)}$

$$\left[\frac{i}{b}A_{-k}, e^{2b\phi(z)}\right] = z^{-k}e^{2b\phi(z)}, \quad \left[\frac{1}{ib}B_k, e^{2b\phi(z)}\right] = z^k e^{2b\phi(z)}$$

Operators A_{-k} commute with V_α and $[\frac{1}{ib}B_k, V_\alpha(1)] = \frac{2\alpha-Q}{b}V_\alpha(1)$.

The calculation of matrix element reduces to the calculation of the Selberg average:

$$\frac{{}_{\mu,\varnothing}\langle P'|V_{\alpha_n}|P\rangle_{\lambda,\varnothing}}{\langle P'|V_{\alpha_n}|P\rangle} = \Omega_\mu(P')\Omega_\lambda(P)\frac{\left\langle \mathbf{J}^{(1/b^2)}[p_k+\rho]\mathbf{J}^{(1/b^2)}[p_{-k}]\right\rangle^{(n)}_{Sel}}{\langle 1\rangle^{(n)}_{Sel}}$$

where $\rho = \frac{2\alpha-Q}{b}$ and $\langle \mathcal{O}_n\rangle^{(n)}_{Sel}$ denotes:

$$\frac{1}{n!}\int_0^1 .. \int_0^1 \mathcal{O}(t_1, .., t_n) \prod_{i<k}^{n} |t_i - t_k|^{-2b^2} \prod_{j=1}^{n} t_j^A (1-t_j)^B dt_j$$

with parameters $A = -b(Q+2P), B = -2b\alpha_n$. This integral can be calculated and gives the expected result for matrix element.

- We note that $\Omega_\lambda(P)$ vanishes for

$$P = P_{m,n} = -\frac{mb + nb^{-1}}{2}, \qquad \text{for} \quad (m,n) \in \lambda$$

- At $P = P_{m,n}$ the Verma module $|P\rangle$ is degenerate, i.e. there exists a singular vector $|\chi_{m,n}\rangle$ at the level mn

$$|\chi_{m,n}\rangle \overset{\text{def}}{=} D_{m,n}|P_{m,n}\rangle = \left(L_{-1}^{mn} + \dots\right)|P_{m,n}\rangle,$$

such that $L_k|\chi_{m,n}\rangle$ for any $k > 0$.

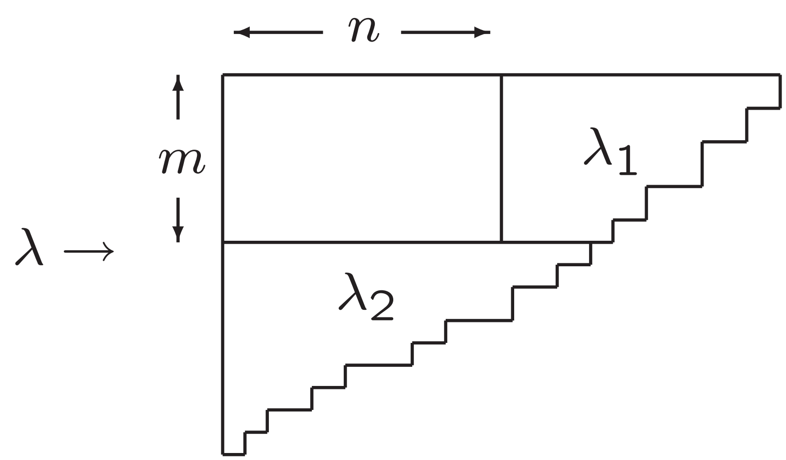
Proposition: Let us define the operator $X_{\vec{\lambda}}(P) = X_{\lambda_1,\lambda_2}(P)$ as

$$X_{\vec{\lambda}}(P)|P\rangle \overset{\text{def}}{=} |P\rangle_{\vec{\lambda}},$$

then the following relation holds

$$X_{\lambda,\varnothing}(P_{m,n})|P_{m,n}\rangle = (-1)^{mn} X_{\lambda_1,\lambda_2}(P_{m,-n})\, D_{m,n}|P_{m,n}\rangle \quad \text{for} \quad (m,n) \in \lambda,$$

where the pair of Young diagrams (λ_1, λ_2) is defined by the following "cutting" rule



This equation can be considered as a definition of $X_{\lambda_1,\lambda_2}(P)$ at $P = P_{m,-n}$.

Integrals of Motion and Classical Limit

One can check that the states $|P\rangle_{\vec{\lambda}}$ are the eigenstates of the following infinite system of the Integrals of Motion

$$\mathbf{I}_2 = L_0 - \frac{c}{24} + 2\sum_{k=1}^{\infty} a_{-k}a_k,$$

$$\mathbf{I}_3 = \sum_{k=-\infty,k\neq 0}^{\infty} a_{-k}L_k + 2iQ\sum_{k=1}^{\infty} k a_{-k}a_k + \frac{1}{3}\sum_{i+j+k=0} a_i a_j a_k,$$

$$\mathbf{I}_4 = 2\sum_{k=1}^{\infty} L_{-k}L_k + L_0^2 - \frac{c+2}{12}L_0 + 6\sum_{k=-\infty,k\neq 0}^{\infty}\sum_{i+j=k} L_{-k}a_i a_j +$$
$$+12\left(L_0 - \frac{c}{24}\right)\sum_{k=1}^{\infty} a_{-k}a_k + 6iQ\sum_{k=-\infty,k\neq 0}^{\infty} |k|\, a_{-k}L_k +$$
$$+2(1-5Q^2)\sum_{k=1}^{\infty} k^2 a_{-k}a_k + 6iQ\sum_{i+j+k=0} |k|\, a_i a_j a_k + \sum_{i+j+k+l=0} :a_i a_j a_k a_l:$$

....

- In semiclassical limit $b \to 0$

$$T \to -Q^2 u, \qquad \partial\varphi \to -Qv, \qquad [\ ,\ ] \to -\frac{2i\pi}{Q^2}\{\ ,\ \},$$

we find that u and v satisfy Poisson bracket algebra relations

$$\{u(x), u(y)\} = (u(x) + u(y))\,\delta'(x-y) + \frac{1}{2}\,\delta'''(x-y),$$
$$\{v(x), v(y)\} = \frac{1}{2}\delta'(x-y), \quad \{u(x), v(y)\} = 0.$$

One can recover classical Hamiltonian system taking $\mathcal{H} = \int G_3(y)dy$

$$G_3 = uv + v\mathrm{D}v + \frac{1}{3}v^3,$$

where $\mathrm{D} = \frac{d}{dx}\mathrm{H}$ and H is the operator of Hilbert transform defined by the principal value integral

$$\mathrm{H}\,F(x) \stackrel{\mathrm{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} F(y)\cot\frac{1}{2}(y-x)\,dy,$$

- So, we defined integrable system of equations

$$\begin{cases} u_t + vu_x + 2uv_x + \frac{1}{2}v_{xxx} = 0, \\ v_t + \frac{u_x}{2} + \mathsf{H}v_{xx} + vv_x = 0, \end{cases}$$

- It possesses infinitely many conserved quantities $I_k = \int G_k dx$

$$G_2 = u + v^2,$$
$$G_3 = uv + v\mathsf{D}v + \frac{1}{3}v^3,$$
$$G_4 = u^2 + 6uv^2 + 6u\mathsf{D}v + 5v_x^2 + 6v^2\mathsf{D}v + v^4,$$
$$G_5 = u^2v + \frac{1}{2}u\mathsf{D}u + 2u_xv_x + 4uv\mathsf{D}v + v^2\mathsf{D}u + 2uv^3 + \frac{3}{2}v_x\mathsf{D}v_x + \\ + 3\,vv_x^2 + 2v(\mathsf{D}v)^2 + \frac{4}{3}v^3\mathsf{D}v + \frac{1}{2}v^2\mathsf{D}v^2 + \frac{1}{5}v^5,$$

....

- It is convenient to represent $u = w^2 - w_x$ and define $\psi = v + iw$

$$\psi_t + \frac{i}{2}\psi^*_{xx} + \psi\psi_x + H\,\mathrm{Re}\,\psi_{xx} = 0,$$

Nekrasov Part. Funct. and Zamolodchikov's Rec. Rel.

One-point conformal block $\mathcal{F}_\alpha^{(\Delta)}(q)$ is defined as the contribution to the trace of the conformal family with conformal dimension $\Delta = \frac{Q^2}{4} + P^2$

$$\mathcal{F}_\alpha^{(\Delta)}(q) \overset{\text{def}}{=} \text{Tr}_\Delta \left(q^{L_0 - \frac{c}{24}} V_\alpha(0) \right) = 1 + \frac{2\Delta + \Delta^2(\alpha) - \Delta(\alpha)}{2\Delta} q + \ldots$$

It was proposed by Alday, Gaiotto and Tachikawa that

$$\mathcal{F}_\alpha^{(\Delta)}(q) = \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)} \right)^{2\Delta(\alpha)-1} Z(\varepsilon_1, \varepsilon_2, m, a),$$

where $Z(\varepsilon_1, \varepsilon_2, m, a)$ is the instanton part of the Nekrasov partition function in $\mathcal{N} = 2^*$ $U(2)$ SYM with

$$P = \frac{a}{\hbar}, \qquad \alpha = \frac{m}{\hbar}, \qquad \varepsilon_1 = \hbar b, \qquad \varepsilon_2 = \frac{\hbar}{b},$$

where a is VEV of scalar field, m is the mass of the adjoint hypermultiplet and ε_k are the parameters of the Ω background. Parameter q is given by

$$q = e^{2i\pi\tau}, \quad \text{where} \quad \tau = \frac{4i\pi}{g^2} + \frac{\theta}{2\pi}.$$

Nekrasov partition function

$$Z(\varepsilon_1, \varepsilon_2, m, a) = 1 + \sum_{k=1}^{\infty} q^k \mathfrak{z}_k,$$

can be represented as a sum over partitions. Let $\vec{\nu} = (\nu_1, \nu_2)$ be the pair of Young diagrams with the total numbers of cells equal to N. Then

$$\mathfrak{z}_N = \sum_{\vec{\nu}} \prod_{i,j=1}^{2} \prod_{s \in \nu_i} \frac{(E_{ij}(s) - \alpha)(Q - E_{ij}(s) - \alpha)}{E_{ij}(s)(Q - E_{ij}(s))},$$

where

$$E_{ij}(s) = 2P\epsilon_{ij} - b l_{\nu_j}(s) + b^{-1}(a_{\nu_i}(s) + 1),$$

$a_\nu(s)$ and $l_\nu(s)$ are respectively the horizontal and vertical distance from the square s to the edge of the diagram ν.

- AGT relation for $N = 2^*$ theory can proved using Al. Zamolodchikov's recursive formula

- The coefficient $\mathfrak{Z}_N$ can be represented as the contour integral

$$\mathfrak{Z}_N = \frac{1}{N!}\left(\frac{Q(b-\alpha)(b^{-1}-\alpha)}{2\pi\mathrm{i}\alpha(Q-\alpha)}\right)^N \oint_{\mathcal{C}_1}\ldots\oint_{\mathcal{C}_N}\prod_{k=1}^{N}\frac{\mathcal{P}(x_k+\alpha)\mathcal{P}(x_k+Q-\alpha)}{\mathcal{P}(x_k)\mathcal{P}(x_k+Q)}\times$$

$$\times\prod_{i<j}\frac{x_{ij}^2(x_{ij}^2-Q^2)(x_{ij}^2-(b-\alpha)^2)(x_{ij}^2-(b^{-1}-\alpha)^2)}{(x_{ij}^2-b^2)(x_{ij}^2-b^{-2})(x_{ij}^2-\alpha^2)(x_{ij}^2-(Q-\alpha)^2)}\,dx_1\ldots dx_N,$$

where $\mathcal{P}(x)=(x-P_1)(x-P_2)$ with $P=(P_1-P_2)/2$. The contour $\mathcal{C}_k$ surrounds poles $x_k=P_1$, $x_k=P_2$, $x_k=x_j+b$ and $x_k=x_j+b^{-1}$.

- A singularity in $\mathfrak{Z}_N=\mathfrak{Z}_N(\alpha,\Delta)$ ($\Delta=Q^2/4+P^2$) can happen when two poles of the integrand pinch the contour. One can show that

$$\mathrm{Res}\,\mathfrak{Z}_N(\alpha,\Delta)\Big|_{\Delta=\Delta_{m,n}} = R_{m,n}(\alpha)\,\mathfrak{Z}_{N-mn}(\alpha,\Delta_{m,-n}),$$

where $R_{m,n}(\alpha)$ is exactly the same as prescribed by Alyosha Zamolodchikov's recursion formula.

- So, the singular part of the Nekrasov partition function coincides with the singular part of the one-point conformal block.

- The non-singular part which can be obtained in the limit $\Delta \to \infty$. It can be found using well known "hook-length" formula

$$\left(\frac{q^{\frac{1}{24}}}{\eta(\tau)}\right)^{1-\lambda} = 1 + \sum_{k=1}^{\infty} \xi_k(\lambda) q^k,$$

with

$$\xi_N(\lambda) = \sum_{\nu} \prod_{s\in\nu} \left(1 - \frac{\lambda}{\big(1 + l_\nu(s) + a_\nu(s)\big)^2}\right).$$

the sum goes over all ν's with the total number of cells equal to N.

- Comparing asymptotics of the conformal block and Nekrasov partition function one finds the coefficient of proportionality in AGT formula.

- Seiberg-Witten prepotential can be obtained in the semiclassical limit $\hbar \to 0$

$$Z(\varepsilon_1, \varepsilon_2, m, \vec{a}) \to \exp\left(\frac{1}{\hbar^2}\mathcal{F}\left(m, \vec{a}|q\right) + O(1)\right).$$

- To derive this limit from the Liouville point of view we consider two-point function with one degenerate field

$$\Psi(z) \sim \langle V_{-\frac{b}{2}}(z) V_\alpha(0)\rangle$$

This function satisfies Scrödinger equation

$$\left(-\partial_z^2 + \frac{b^2 m^2}{\hbar^2}\wp(z)\right)\Psi(z) = \frac{2ib^2}{\pi}\partial_\tau \Psi(z).$$

- We look for the solution in the form

$$\Psi(z) = \exp\left(\frac{1}{\hbar^2}\mathcal{F}(q) + \frac{b}{\hbar}\mathcal{W}(z|q) + \dots\right)$$

with prescribed monodromy $e^{2i\pi a}$ around A-cycle.

- WKB approximation gives

$$\mathcal{W}(z|q) = \int \sqrt{E(q) + m^2 \wp(z)} dz, \qquad E(q) = 4q\partial_q \mathcal{F}(q).$$

- With $E(q)$ given in parametric form

$$\oint_A \sqrt{E(q) + m^2 \wp(z)} dz = 2i\pi a,$$

the prepotential $\mathbb{F}(m, \vec{a}|q)$ can be calculated as follows

$$\mathbb{F}(m, \vec{a}|q) = \left(a^2 + \frac{m^2}{12}\right) \log(q) - 4m^2 \log(\eta(\tau)) + \mathcal{F}(q),$$

- The integral over B-cycle defines a_D

$$\oint_B \sqrt{E(q) + m^2 \, \wp(z)} \, dz = 2i\pi a_D,$$

which is the derivative of the total prepotential (including classical and perturbative part) with respect to a.

Conformal Toda Theory (review)

$sl(n)$ CTT is described by the density of Lagrangian:

$$\frac{1}{2\pi}\partial\phi\bar{\partial}\phi + \kappa \sum_{i=1}^{n-1} e^{e_i \cdot \phi}$$

where $\phi = (\phi_1, .., \phi_{n-1})$ and e_i are the simple roots of $sl(n), \partial = \partial_z$ CTT possesses $W-$ symmetry, generated by currents $\mathsf{W}_k(z)$. Let h_i are the weights of the first fundamental representation with h.w. $\omega_1 : h_k = \omega_1 - \sum_1^{k-1} e_i$. We define the Miura transformation: $\phi \to \mathsf{W}$

$$\prod_{i=0}^{n} (Q\partial + h_{n-i} \cdot \partial\phi) = \sum_{k=0}^{n} \mathsf{W}_{n-k}(z)\,(Q\partial)^k$$

which establishes the representation for currents $\mathsf{W}_k(z)$. $\mathsf{W}_0 = 1, \mathsf{W}_2 = T(z)$. The primary fields of $W-$algebra are the exponential Toda fields $V_\alpha(z)$:

$$V_\alpha = e^{\alpha \cdot \phi}$$

$$\mathsf{W}_k(z)V_\alpha(z') = \frac{w_k(\alpha)}{(z-z')^k}V_\alpha(z') + O((z-z')^{-k+1})$$

The Weyl invariant functions $w_k(\alpha)$ determine h.w. representation. We introduce Weyl vector $\rho-$half of the sum of positive roots and define vector $P = \alpha - Q\rho$. Let parameters $x_i = h_i \cdot P$,then $w_2 = \Delta$ and w_3 are:

$$\Delta = \frac{1}{2}((Q\rho)^2 - \sum_1^n x_i^2), \quad w_3 = \frac{1}{3}\sum_1^n x_i^3.$$

$W-$symmetry together with three point functions of primary fields do not fix in general conformal blocks. However, they fix completely the blocks:

$$\left\langle V_{\alpha_1}(z_1)V_{a_2\omega_1}(z_2)V_{a_2\omega_1}(z_3)...V_{a_{k-1}\omega_1}(z_{k-1})V_{\alpha_k}(z_k)\right\rangle$$

According to the AGT-Wyllard conjecture this block coincides with instanton contribution of $k-3$ $SU(n)$ gauge theories coupled with n fundamental n anti-fundamental and $k-4$ bifundamental matter super multiplets.

Integrals of Motion and Matrix Elements

$$\mathbf{I}_1^{(n)} = L_0 + \sum a_{-k} a_k$$

$$\mathbf{I}_3^{(n)} = i\frac{n^{3/2}}{2} Q \sum_{k=1}^{\infty} a_{-k} a_k + 2 \sum_{k=-\infty}^{\infty} a_{-k} L_k + \frac{1}{3} \sum_{i+j+k=0} a_i a_j a_k + n^{1/2} W_3^{(0)}$$

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The eigenvectors $\Psi_{\vec{\nu},P}$ of $\mathbf{I}_3^{(n)}$ are parametrized by n partitions $\vec{\nu} = (\nu_1, .., \nu_n)$, and corresponding eigenvalues are:

$$\mathcal{I}_3 = \sum_{i=1}^{n} q_i^{(3)}(x_i, \nu_i)$$

where $q^{(3)}(x,\nu) = i(-2|\nu|x + \frac{1}{b}\sum_l \nu_l(\nu_l + 2(l-1)b^2))$.

The similar property of decomposition is valid for eigenvalues $\mathcal{I}_k$ of all integrals $\mathbf{I}_k^{(n)}$

The matrix elements of vertex operator

$$\mathcal{V}_a(z) = e^{\sqrt{n}a\varphi_+} e^{-\sqrt{n}(Q-a)\varphi_-} V_{a\omega_1}(z)$$

in the basis of these vectors $F^{\nu}_{\nu'}(a,P,P') = \dfrac{\left\langle \Psi_{\vec{\nu},P} | \mathcal{V}_a(1) | \Psi_{\vec{\nu}',P'} \right\rangle}{\left\langle \Psi_{\vec{0},P} | \mathcal{V}_a(1) | \Psi_{\vec{0},P'} \right\rangle}$ are equal

$$F^{\vec{\nu}}_{\vec{\nu}'}(a,P,P') = \prod_{i,j=1}^{n} \prod_{s\in\nu'_i} (Q - E_{\nu'_i,\nu_j}(x_j - x'_i|s) - a) \prod_{t\in\nu_j} (E_{\nu_j,\nu'_i}(x'_i - x_j|t) - a)$$

This expression coincides with Z_{bif} derived by Nekrasov in instanton calculations.

The norms $N_{\vec{\nu}}(P)$ of the vectors $\Psi_{\vec{\nu},P}$ are equal to $F^{\vec{\nu}}_{\vec{\nu}}(0,P,P)$:

$$N_{\vec{\nu}}(P) = \prod_{i,j=1}^{n} \prod_{s\in\nu_i} (Q - E_{\nu_i,\nu_j}(x_j - x_i|s)) \prod_{t\in\nu_j} (E_{\nu_j,\nu_i}(x_i - x_j|t))$$

and coincides with Z^{-1}_{vec}

Classical Integrable System

It is convenient to introduce the fields

$$\phi_j = \frac{1}{\sqrt{n}}(\varphi + h_j \cdot \phi)$$

which in the classical limit have the canonical Poisson brackets:

$$\left\{\phi_i(x), \phi_j(y)\right\} = \delta_{i,j}\delta'(x-y).$$

In terms of these fields two first classical densities of integrals have a form:

$$G_2^{(n)} = \sum_{k=1}^{n} \phi_k^2$$

$$G_3^{(n)} = \frac{1}{2i}\widehat{\phi}\mathsf{D}\widehat{\phi} + \sum_{j>k}^{n} \phi_j \partial \phi_k - \frac{1}{3}\sum_{k=1}^{n} \phi_k^3$$

where $\widehat{\phi} = \sum_{j=1}^{n} \phi_j$, and D is derivative of Hilbert transform on the circle.

$$\mathsf{H}\, F(x) \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} F(y) \cot \frac{1}{2}(y-x)\, dy,$$

Let

$$\sigma[i] = 1 \quad if \quad i > 0, \quad 0 \quad if \quad i = 0, \quad -1 \quad if \quad i < 0.$$

Then equations of motion with Hamiltonian $I_3^{(n)}$ can be written as:

$$\frac{1}{i}\partial_t\phi_j + \frac{1}{i}\mathsf{D}\widehat{\phi} + \sum_{k=1}^{n} \sigma[j-k]\partial_x^2\phi_k - 2\phi_j\partial_x\phi_j = 0$$

These equations admit the "reality" conditions $\phi_k^* = -\phi_{n+1-k}$

In the limit $n \to \infty$ we can define the variable $y = \frac{j}{n}$, denote $\phi_j = n\chi_{\frac{j}{n}}$ and introduce the function $u(x,y,t) = \chi_y(x,nt)$ which satisfies the equations:

$$\frac{1}{i}\partial_t\partial_y u + 2\partial_x^2 u - \partial_x\partial_y u^2 = 0,$$

$$\frac{1}{i}\partial_t u(x,0,t) + \int_0^1 (\frac{1}{i}\mathsf{D}u(x,y,t)_x - u(x,y,t)_{xx})dy - \partial_x u^2(x,0,t) = 0$$

These equations have a stationary solution (soliton)

$$u(x,y) = (1/2 - y)\cot(x/2 + sign(y - 1/2)i\eta)$$

This solution is large n limit of of smooth in j solution of system of equations for ϕ_j

$$\phi_j = (\frac{n+1}{2} - j)\cot(x/2 + sign(j - \frac{n+1}{2})i\eta)$$