Unusual singular behaviour of the Entanglement Entropy in 1D systems

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Collaboration with

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Outline

- Introduction: Von Neumann and Renyi entropies as a measure of Entanglement
- Entanglement entropy in 1D lattice spin chains: the Corner Transfer Matrix (CTM) method
- XYZ chain exact Entanglement Entropy
- Essential critical point for the entropy
- Quantum Entropy in integrable models
- Conclusions

Introduction

Entanglement: fundamental quantum property

Different reasons for interest:

1. Quantum Information $\longrightarrow$ Quantum computers
2. Quantum Phase Transitions $\longrightarrow$ universality
3. Condensed matter physics $\longrightarrow$ non-local correlators
4. Integrable models $\longrightarrow$ new playground
5. Black holes $\longrightarrow$ Information paradox & Quantum Gravity
6. **NON LOCALITY** intrinsic in Quantum Mechanics?

- **EPR paradox**: uncompleteness of QM or non-locality?
- **Bell inequalities** $\longrightarrow$ local hidden variables exist only if $\mathcal{P} < 2$
- Clauser Friedmann & Aspect **experiments** $\longrightarrow \mathcal{P} > 2 \Longrightarrow$ possible non-locality of QM

Quantum systems and sub-systems

- Quantum system with unique pure ground state $|0\rangle$ composed of two subsystems, A and B.

- If a state has wavefunction

$$|\psi^{A\otimes B}\rangle = |\psi^A\rangle \otimes |\psi^B\rangle$$

Separable $\Longrightarrow$ No entanglement
(i.e. measurements on B do not affect A state)

- If instead

$$|\psi^{A\otimes B}\rangle = \sum_{j=1}^{d} \lambda_j |\psi_j^A\rangle \otimes |\psi_j^B\rangle$$

with $d > 1 \Longrightarrow$ **Entangled** (measurements of B do affect A state)

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Density matrix and mixed states

States can be represented by **kets** or by **density matrices** (Von Neumann 1927)

- Density Matrix of **pure state** $|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi|$$

- Ensemble $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle$ of states prepared each with probability $p_1, p_2, ..., p_n$

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  leads to the concept of **mixed state**

- Schrödinger equation can be recast in terms of density matrix as

$$i\hbar \frac{\partial}{\partial t}\rho = [H, \rho]$$

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- Define **reduced density matrix** for subsystem A

$$\rho_A = \mathrm{Tr}_B \rho$$

Quantum entropy (Von Neumann) of Entanglement (E-Entropy)

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A) = S_B$$

(Bennett, Bernstein, Popescu, Schumacher 1996)

For a separable state $S_A = 0$, for a maximally entangled state it is maximal $\Longrightarrow$ S_A is a measure of Entanglement

How to measure entanglement

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Renyi Entropy

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$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{Tr}_A \rho_A^\alpha$$

Introduced by hungarian mathematician Rényi in probability theory

- It reduces to Von Neumann for $\alpha \to 1$
- Contains higher momenta and for $\alpha \to \infty$ the spectrum of the reduced density matrix ρ_A can be read
- link with replica trick à la Calabrese Cardy

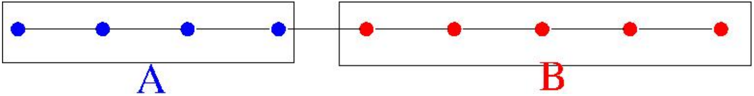
Entanglement in a Spin Chain

- Hamiltonian

$$H = \sum_{k=1}^{N} H_{k,k+1}$$

- Consider the ground state with $\rho = |0\rangle\langle 0|$
- Block of spins in the space interval $[1, \ell]$ is subsystem A
- The rest of the ground state is subsystem B

$\Longrightarrow$ Entanglement of a block of spins in the space interval $[1, \ell]$ with the rest of the ground state as a function of ℓ



General Behavior (Area Law)

- Asymptotic behaviour (block size $\ell \to \infty$) in a double scaling limit $0 \ll \ell \ll N$ of von Neumann E-Entropy

$$S(\ell) = -\text{Tr}_A(\rho_A \log \rho_A)$$

- For gapped phases (Vidal, Latorre, Rico, Kitaev 2003)

$$S(\ell) \simeq \text{const.} + \ldots$$

- For critical conformal phases (Calabrese, Cardy 2004)

$$S(\ell) \simeq \frac{c}{3} \log \ell + \ldots$$

c = central charge of CFT

E-Entropy and Universality

- Powers of ρ easily accessible in CFT (replica trick)

$$S_\alpha(\ell) = \frac{1}{1-\alpha} \mathrm{Tr}_A(\rho_A^\alpha)$$

- h = scaling dimension of the operator responsible for the correction (Calabrese, Cardy 2004-2010)

$$S_\alpha(\ell) = \frac{c}{6}\left(1 - \frac{1}{\alpha}\right)\log \ell + \overbrace{c'_\alpha + \underbrace{b_\alpha(\ell)\ell^{-\frac{2h}{\alpha}} + \dots}}^{non-universal}$$

Conjectured

- Close to criticality $\xi \sim a^{-1}$ and $\ell \to \infty$

$$S_\alpha = \frac{c}{6}\left(1 - \frac{1}{\alpha}\right)\log \frac{\xi}{a} + \overbrace{C'_\alpha + B_\alpha \left(\frac{\xi}{a}\right)^{-\frac{2x}{\alpha}} + \dots}$$

(Calabrese, Cardy, Peschel 2010)

XYZ model

Hamiltonian

$$H_{XYZ} = - J \sum_{k=1}^{N} (\sigma_k^x \sigma_{k+1}^x + J_y \sigma_k^y \sigma_{k+1}^y + J_z \sigma_k^z \sigma_{k+1}^z)$$

commutes with transfer matrix of 8-vertex model

- for $J_y = 1$ it gives XXZ model
  - for $J_y = J_z = 1$ ferromagnetic XXX
  - for $J_y = 1,\ J_z = -1$ antiferromagnetic XXX
- it can be seen as a particularly interesting lattice regularization of the sine-Gordon model

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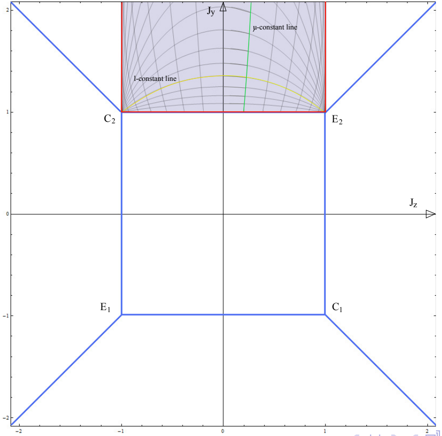
Approaching criticality the Calabese - Cardy (2004) formula holds

$$S_A = \frac{c}{6} \log \frac{\xi}{a} + \mathrm{cost.}$$

everywhere but at the $E_{1,2}$ points



XYZ chain - Parametrization

Symmetries allow to remap H_{XYZ} into a single region of $J_y, J_z \longrightarrow \Gamma, \Delta$

$$H_{XYZ} = -J \sum_{k=1}^{N} (\sigma_k^x \sigma_{k+1}^x + \Gamma \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z)$$

- $|J_y| \leq 1$ and $J_z \leq -1$: $J_y = \Gamma$ and $J_z = \Delta$
- $J_y \geq 1$ and $|J_z| \leq 1$: $J_y = -\Delta$ and $J_z = -\Gamma$
- $|J_y| < 1$ and $|J_z| < 1$: $\Gamma = \frac{|J_z - J_y| - |J_z + J_y|}{|J_z - J_y| + |J_z + J_y|}$ and $\Delta = -\frac{2}{|J_z - J_y| + |J_z + J_y|}$
- $|J_y| > 1$ or $|J_z| > 1$:

$$\Gamma = \frac{\min\left[1, \left|\frac{|J_z - J_y| - |J_z + J_y|}{2}\right|\right]}{\max\left[1, \left|\frac{|J_z - J_y| - |J_z + J_y|}{2}\right|\right]} \cdot \mathrm{sgn}\left[|J_z - J_y| - |J_z + J_y|\right]$$

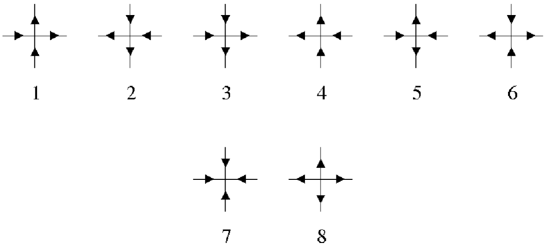
$$\Delta = -\frac{1}{2} \frac{|J_z - J_y| + |J_z + J_y|}{\max\left[1, \left|\frac{|J_z - J_y| - |J_z + J_y|}{2}\right|\right]}$$

XYZ model and 8-vertex model

- XYZ is the hamiltonian limit of 8-vertex model, with partition function

$$Z = \sum \prod_{i=1}^{8} w_i^{n_i}$$

where the 8 Boltzmann weights $w_i = e^{-\beta\epsilon_i}$ appear n_i times each on the lattice.



$$w_1 = w_2 = a, \; w_3 = w_4 = b, \; w_5 = w_6 = c, \; w_7 = w_8 = d$$

Elliptic parametrization

- A convenient parametrization of the Boltzmann weights

$$\begin{aligned} a &= \rho\, \mathrm{sn}(i\lambda - iu) \\ b &= \rho\, \mathrm{sn}(iu) \\ c &= \rho\, \mathrm{sn}(i\lambda) \\ d &= \rho\, k\, \mathrm{sn}(i\lambda)\, \mathrm{sn}(iu)\, \mathrm{sn}(i\lambda - iu) \end{aligned}$$

- In this parametrization $0 < k < 1$ and $0 \leq \lambda \leq I(k')$

$$J_z = -\Gamma = -\frac{1 + k^2 \mathrm{sn}^2(i\lambda)}{1 - k^2 \mathrm{sn}^2(i\lambda)} \quad , \quad J_y = -\Delta = -\frac{\mathrm{cn}(i\lambda)\, \mathrm{dn}(i\lambda)}{1 - k^2 \mathrm{sn}^2(i\lambda)}$$

More convenient

$$l = \frac{2\sqrt{k}}{1+k} \quad , \quad \mu = \frac{\pi\lambda}{I(k')}$$

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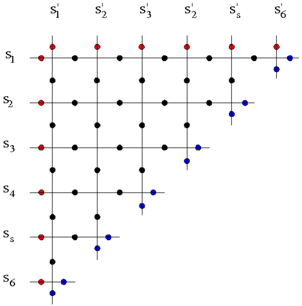
Corner Transfer Matrix of 8-vertex

- CTM is a very useful tool introduced by Baxter (1972)

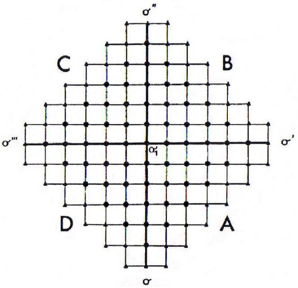


$$A_{\bar{s},\bar{s}'} = \sum_{\bullet} \prod w_i$$

- and analogously B, C, D with 90° rotations. One can prove that $A = C$ and $B = D$.

Partition function and CTM

- Define $B_{\bar{\sigma}\bar{\sigma}'}$ in the same way as $A_{\bar{\sigma}\bar{\sigma}'}$ only with the last figure rotated anticlockwise by 90°. Similarly define $C_{\bar{\sigma}\bar{\sigma}'}$ and $D_{\bar{\sigma}\bar{\sigma}'}$ by rotating by 180° and 270°.
- Now we can build up the whole lattice by using the 4 CTM's



- Partition function

$$\mathcal{Z} = \sum_{\bar{\sigma},\bar{\sigma}',\bar{\sigma}'',\bar{\sigma}'''} A_{\bar{\sigma}\bar{\sigma}'} B_{\bar{\sigma}'\bar{\sigma}''} C_{\bar{\sigma}''\bar{\sigma}'''} D_{\bar{\sigma}'''\bar{\sigma}} = \mathrm{Tr}(ABCD)$$

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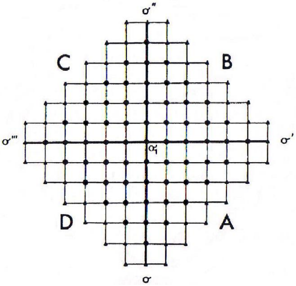

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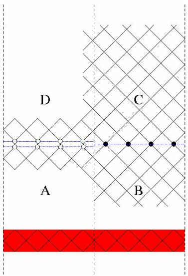
- Now suppose to divide the spins in two subsystems A: $\bar{\sigma}_A = (\sigma_1, ..., \sigma_p)$ and B: $\bar{\sigma}_B = (\sigma_{p+1}, ..., \sigma_L)$, i.e. $\bar{\sigma} = (\bar{\sigma}_A, \bar{\sigma}_B)$
- Reduced density matrix of subsystem A

$$\rho_A(\bar{\sigma}_A, \bar{\sigma}'_A) = \sum_{\bar{\sigma}_B} \langle \bar{\sigma}_A, \bar{\sigma}_B | 0 \rangle \langle 0 | \bar{\sigma}'_A, \bar{\sigma}_B \rangle = \mathrm{Tr}_B \langle \bar{\sigma}_A | 0 \rangle \langle 0 | \bar{\sigma}'_A \rangle$$



Reduced density matrix and EE

- The unnormalized reduced density matrix is

$$\hat{\rho}_A = (ABCD)_{\bar{\sigma},\bar{\sigma}'}$$

Normalization by dividing by the trace

$$\rho_A = \frac{\hat{\rho}_A}{\mathrm{Tr}_A \hat{\rho}_A} = \frac{\hat{\rho}_A}{\mathcal{Z}}$$

- Renyi entropy

$$S_A(\alpha) = \frac{1}{1-\alpha} \mathrm{Tr}_A \rho_A^\alpha$$

- Entanglement entropy

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A = -\frac{\mathrm{Tr}_A \hat{\rho}_A \log \hat{\rho}_A}{\mathrm{Tr}_A \hat{\rho}_A} + \mathrm{Tr}_A \hat{\rho}_A$$

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Diagonalization of CTM

- In the thermodynamic limit Baxter (1977) proved the following formula for the diagonalized CTM

$$
\begin{aligned}
A_d(u) &= C_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & s^3 \end{pmatrix} \otimes \dots \\
B_d(u) &= D_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & t^3 \end{pmatrix} \otimes \dots
\end{aligned}
$$

where

$$
s = \exp\left(-\frac{\pi u}{2I(k)}\right) \qquad , \qquad t = \exp\left(-\frac{\pi(\lambda - u)}{2I(k)}\right)
$$

and $I(k)$ is the elliptic integral of I kind of modulus k

Reduced density matrix

- Define $x = (st)^2 = \exp\left(-\frac{\pi\lambda}{I(k)}\right) = e^{i\mu\tau}$ where $\tau = i\frac{I(k')}{2I(k)}$ ($k' = \sqrt{1-k^2}$) and use the CTM density matrix formula

$$\rho_A = ABCD = (AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^3 \end{pmatrix} \otimes \ldots$$

- $\rho = e^{\epsilon\mathcal{O}}$ where $\mathcal{O}$ is a operator with integer spectrum

$$\mathcal{O} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \otimes \ldots$$

$\epsilon = -\frac{\pi\lambda}{I(k)}$ depends on the XYZ parameters through elliptic functions

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Entanglement entropy of XYZ model

The trace of the reduced density matrix

$$\mathcal{Z} = \mathrm{Tr}\rho_A = \prod_{j=1}^{\infty}(1 + x^j) \qquad \text{and} \qquad S_A = -\epsilon \frac{\log \mathcal{Z}}{\partial \epsilon} + \log \mathcal{Z}$$

leads to the final formula for Von Neumann

$$S_A = \epsilon \sum_{j=1}^{\infty} \frac{j}{(1 + e^{j\epsilon})} + \sum_{j=1}^{\infty} \log(1 + e^{-j\epsilon})$$

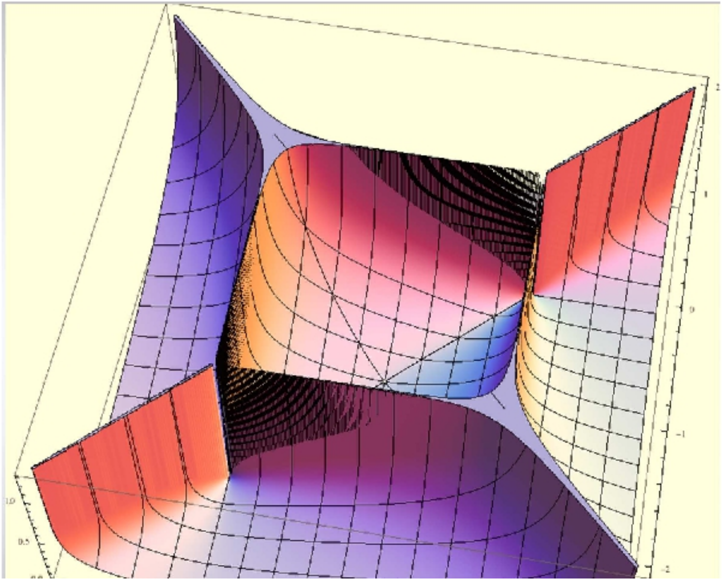
and for Rényi entropy

$$S_\alpha = \frac{\alpha}{\alpha - 1} \sum_{j=1}^{\infty} \log(1 + q^{2j}) + \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \log(1 + q^{2j\alpha})$$

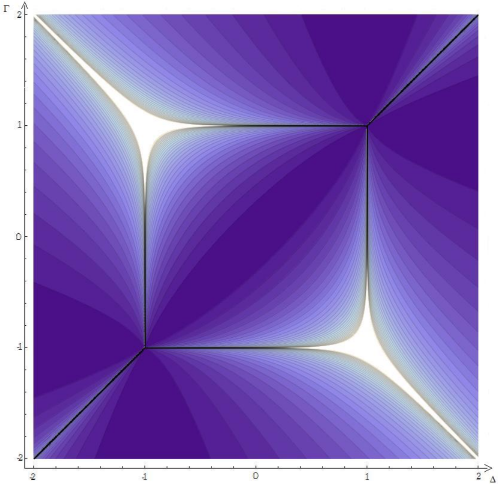
that can also be written in theta function terms

$$S_\alpha = \frac{1}{6(1-\alpha)} \left[\alpha \log \frac{\theta_4(0,q)\theta_3(0,q)}{\theta_2^2(0,q)} + \log \frac{\theta_2^2(0,q^\alpha)}{\theta_3(0,q^\alpha)\theta_4(0,q^\alpha)} \right]$$

Entanglement Entropy 3D plot



Isoentropic lines



Tricritical points

- $C_{1,2}$: conformal points - entropy diverges close to them - linear spectrum
- $E_{1,2}$: Non-conformal points - entropy goes from 0 to ∞ arbitrarily close to them, depending on direction.
  They corrspond to Isotropic ferromagnetic Heisenberg $\longrightarrow$ quadratic spectrum
- Points similar to $E_{1,2}$ previously observed in XY model in magnetic field (Franchini, Its, Korepin)

Conformal points

Expansion close to conformal points $C_{1,2}$ agree with expectations

$$
\begin{aligned}
S_\alpha &= \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\log\xi - \frac{1}{24}\left(11-\frac{1}{\alpha}\right)\log 2 \\
&+ \frac{\alpha}{6(1-\alpha)}\left[\frac{\xi^{-2}}{4}+O(\xi^{-4})\right] \\
&- \frac{1}{6(1-\alpha)}\left[4(4\xi)^{-2/\alpha}+O(\xi^{-4/\alpha})\right]
\end{aligned}
$$

Leading correction $\xi^{-\delta/\alpha}$ with $\delta = 2$. Operator responsible of this correction (Calabrese, Cardy, Peschel - 2010) has conformal dimensions $(\Delta, \bar{\Delta}) = (1,1)$

Non-conformal points

Expanding around E_1:

$$\Gamma = -1 + \delta \cos\phi \qquad , \qquad \Delta = -1 - \delta \sin\phi \qquad \left(0 \leq \phi \leq \frac{\pi}{2}\right)$$

one finds

$$\lambda \sim I(k') \qquad \text{and} \qquad \varepsilon = \frac{I(k')}{I(k)}$$

So ε varies from 0 at $\phi = 0$ to ∞ at $\phi = \frac{\pi}{2}$. Consequently the entropy explores all values from 0 to ∞ approaching E_1 from various directions $\Longrightarrow$ essential singularity.

- Highly symmetric point, higly degenerate ground state $\Longrightarrow$ level crossing, entanglement can change discontinously
- EE can be used as a marker to detect such essential phase transition points
- Cardy-Calabrese formula is non longer valid: what substitues it?

Link with Ising characters

$$\mathcal{Z}(q=x^2)=\prod_{j=1}^{\infty}(1+q^j)=x^{-\frac{1}{12}}\chi_{1,2}^{Ising}(i\epsilon/\pi)$$

and

$$\mathrm{Tr}\hat{\rho}^{\alpha}=\frac{\chi_{1,2}^{Ising}(i\alpha\epsilon/\pi)}{\left[\chi_{1,2}^{Ising}(i\epsilon/\pi)\right]^{\alpha}}$$

Critical XXZ line: approached for $l \to 1$ i.e. $\tau \to 0$ and $x \to 1$. Use modular transformation to get this limit $\tilde{x}=e^{-i\frac{\pi^2}{\mu\tau}}=e^{-\frac{\pi^2}{\epsilon}}$

$$\mathrm{Tr}\hat{\rho}^{\alpha}=2^{\frac{\alpha-1}{2}}\frac{\chi_{1,1}^{Ising}(i\alpha\epsilon/\pi)-\chi_{2,1}^{Ising}(i\alpha\epsilon/\pi)}{\left[\chi_{1,1}^{Ising}(i\epsilon/\pi)-\chi_{2,1}^{Ising}(i\epsilon/\pi)\right]^{\alpha}}$$

Similar to Calabrese, Cardy, Peschel (2011) but with $c=\frac{1}{2}$ characters, not $c=1$ (!)

For $\tilde{x} \to 0$ we get the expansion of S_α near the conformal points

$$S_\alpha = -\frac{1+\alpha}{24\alpha}\log\tilde{x} - \frac{1}{2}\log 2 - \frac{1}{1-\alpha}\sum_{n=1}^{\infty}\sigma(n)\left[\tilde{x}^{\frac{n}{\alpha}} - \alpha\tilde{x}^{n} - \tilde{x}^{\frac{2n}{\alpha}} + \alpha\tilde{x}^{2n}\right]$$

with

$$\sigma(n) = \frac{1}{n}\sum_{\substack{j<k=1 \\ j\cdot k=n}}^{\infty}(j+k) + \sum_{\substack{j=1 \\ j^2=n}}^{\infty}\frac{1}{j}$$

Scaling limit: $\tilde{x} \approx \left(\frac{\xi}{a}\right)^{-2} \approx \left(\frac{\Delta E}{J}\right)^{2}$. Correlation length computed by Johnson, Krinsky, McCoy (1973)

$$a\xi^{-1} = \begin{cases} -\frac{1}{2}\log k_2 & \text{for } 0 \leq \mu \leq \frac{\pi}{2} \\ -\frac{1}{2}\log\frac{k_2}{\mathrm{dn}^2\left(i2I(k_2)\frac{\tau}{\pi}(\mu-\frac{\pi}{2});k_2'\right)} & \text{for } \frac{\pi}{2} < \mu \leq \pi \end{cases} \qquad k_2 = \frac{1-k'}{1+k'}$$

Free excitations $0 \leq \mu \leq \frac{\pi}{2}$

Correlation length formula does not depend on μ. Expanding in $\tilde{x}$

$$\frac{a}{\xi} = 4\tilde{x}^{\frac{1}{2}} + \frac{16}{3}\tilde{x}^{\frac{3}{2}} + \frac{24}{5}\tilde{x}^{\frac{5}{2}} + \ldots$$

and inverting to get $\tilde{x}(\xi)$, then inserting into S_α

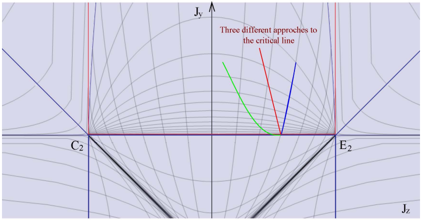
$$\begin{aligned} S_\alpha &= \frac{1+\alpha}{12\alpha}\log\frac{\xi}{a} + \frac{1-2\alpha}{6\alpha}\log 2 \\ &+ B_\alpha \xi^{-\frac{2}{\alpha}} + C_\alpha \xi^{-2\frac{1+\alpha}{\alpha}} + B'_\alpha \xi^{-\frac{4}{\alpha}} + \ldots \\ &- \alpha B_\alpha \xi^{-2} - \alpha B'_\alpha \xi^{-4} + \ldots \end{aligned}$$

for instance

$$B_\alpha = \frac{1}{\alpha - 1}\left(\frac{a}{4}\right)^{\frac{2}{\alpha}}$$

New term not present e.g. in Calabrese Campostrini Essler Nienhuis (2010)

Now ξ depends on μ and one has to specify how to approach the conformal point



1. Renormalization group flow: $\tau = is$, $\mu = \mu_0$ crosses the critcal line with slope $-\frac{2}{\cos\mu_0}$
2. Straight lines in J_y, J_z space: $\tau = -i\frac{\pi}{\log s} + \mathcal{O}(\log^{-2} s)$, $\mu = \mu_0 + \frac{2+m\cos\mu}{2\sin\mu}s + \mathcal{O}(s^2)$
3. Straight lines in l, μ space: $\tau = is$, $\mu = \mu_0 + rs$

Flow analysis (cases 1 & 2)

- **Case 1**: RG flow

$$\frac{a}{\xi} = 4g(\mu_0)\tilde{x}^{\frac{1}{2}} + \frac{16}{3}g^3(\mu_0)\tilde{x}^{\frac{3}{2}} + \mathcal{O}(\tilde{x}^{\frac{5}{2}}) \quad , \quad g(\mu_o) \equiv \cos\frac{\pi^2}{2\mu_0}$$

$$S_\alpha = \frac{1+\alpha}{12\alpha}\log\frac{\xi}{a} + A_\alpha + B_\alpha\left(\frac{\xi}{a}\right)^{-\frac{2}{\alpha}} - \alpha B_\alpha\left(\frac{\xi}{a}\right)^{-2} + C_\alpha\left(\frac{\xi}{a}\right)^{-2-\frac{2}{\alpha}} + \ldots$$

Similar to repulsive case, but coefficients depend on μ_0 and there appears the "new" term

- **Case 2**:

$$S_\alpha = \frac{1+\alpha}{12\alpha}\log\frac{\xi}{a} + A_\alpha(\mu_0) + B_\alpha(\mu_0)\left(\frac{\xi}{a}\right)^{-\frac{2}{\alpha}} + D_\alpha(m,\mu_0)\left(\frac{\xi}{a}\right)^{-\frac{2\mu_0}{\pi}} + \ldots$$

Operator of dimension $h = \frac{\beta^2}{8\pi}$ (Sine-Gordon perturbation field $e^{i\beta\phi}$). Why?

Flow analysis (case 3)

Case 3:

$$S_\alpha = \frac{1+\alpha}{12\alpha}\log\frac{\xi}{a} + A_\alpha(\mu_0) + \frac{E_\alpha(r,\mu_0)}{\log\frac{\xi}{a}}$$

Logarithmic term corresponding to the operator changing the compactification radius at criticality.

Conclusions (1)

- We have got Von Neumann and Rényi EE from integrability in the XYZ spin chain, valid everywhere. It can be written in nice modular form (theta functions) and its modular properties should be investigated further
- Inspecting this formula near critical points, we have discovered essential singularities with unusual critical behaviour. EE can be used as a marker to discriminate behaviours of phase transistion points.
- Corrections to finite subsystem size can be expressed in terms of Ising CFT characters. Maybe interpretable in terms of Majorana fermions η_j construction of the corner transfer matrix (Peschel 2010)

$$\hat{\rho} \propto e^{-H_{CTM}} \qquad \text{with} \qquad H_{CTM} = \sum_{j=1}^{\infty} 2\epsilon_j \eta_j^\dagger \eta_j$$

Conclusions (2)

- For a massive model, the corrections to the entanglement entropy as a function of the correlation length ξ are **different** and require a separate analysis than those yielding the entropy as function of the subsystem size ℓ.

Thank you!!!