Nonperturbative scalar field dynamics in de Sitter space

Julien Serreau, Univ. Paris Diderot

R. Parentani
F. Gautier
M. Guilleux

Motivations

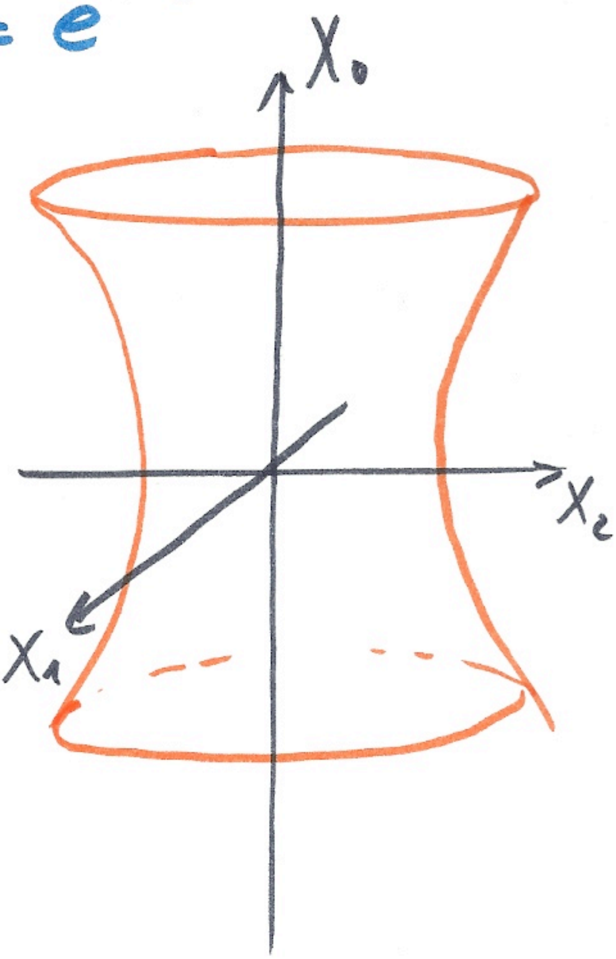
- Radiative corrections to inflationary dynamics
- (Analog) black hole radiation
- Curvature-induced phase transitions
- Foundations of QFT in curved space-times

•
•
•

Scalar fields in de Sitter space (I)

$$ds^2 = -dt^2 + \bar{a}^2(t)\, d\vec{X}^2$$

$$\bar{a}(t) = e^{Ht}$$



$$\boxed{d\eta = dt/\bar{a}(t)}$$

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\vec{X}^2\right)$$

spatially homogeneous
but nonstationary

$$\boxed{\vec{x} = a(t)\vec{X}}$$

$$ds^2 = -(1-x^2)dt^2 - 2\vec{x}\cdot d\vec{x}\,dt + d\vec{x}^2$$

stationary but inhomogeneous

Scalar fields in dS space (II)

$$S = \int d^D x \sqrt{-g(x)} \left(\frac{1}{2} \phi \Box \phi - \frac{m^2}{2} \phi^2 \right)$$

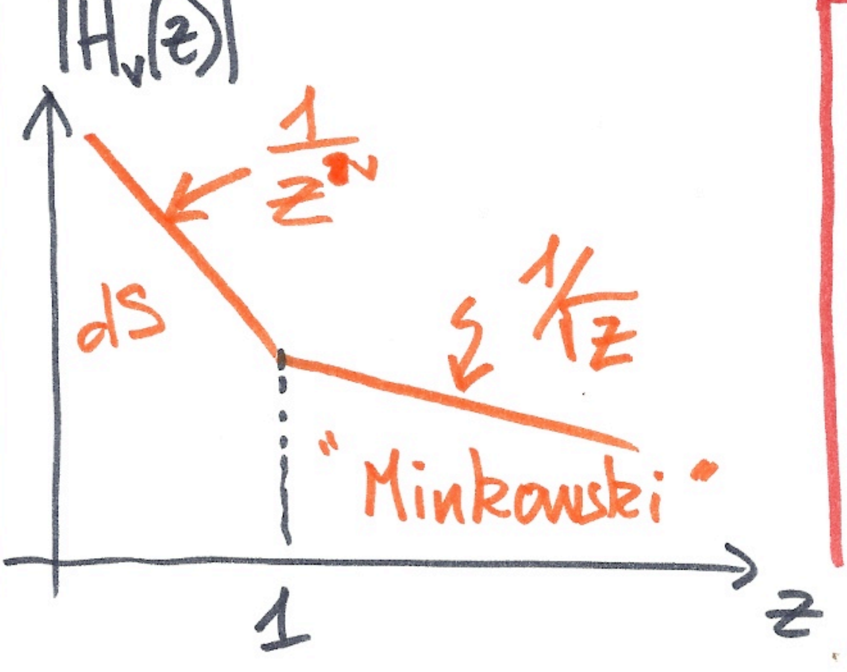
$D = d + 1$; $\Box = \frac{1}{a^2(\eta)} \left(-\partial_\eta^2 + \frac{d-1}{\eta} \partial_\eta + \vec{\nabla}_x^2 \right)$

$$\boxed{(-\Box + m^2)\phi = 0}$$

$$\phi(\eta, \vec{X}) \sim \int \frac{d^d K}{(2\pi)^d} \left(e^{i \vec{K} \cdot \vec{X}} H_\nu\!\left(\frac{K}{a(\eta)}\right) a_K + h.c. \right)$$

$\nu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$

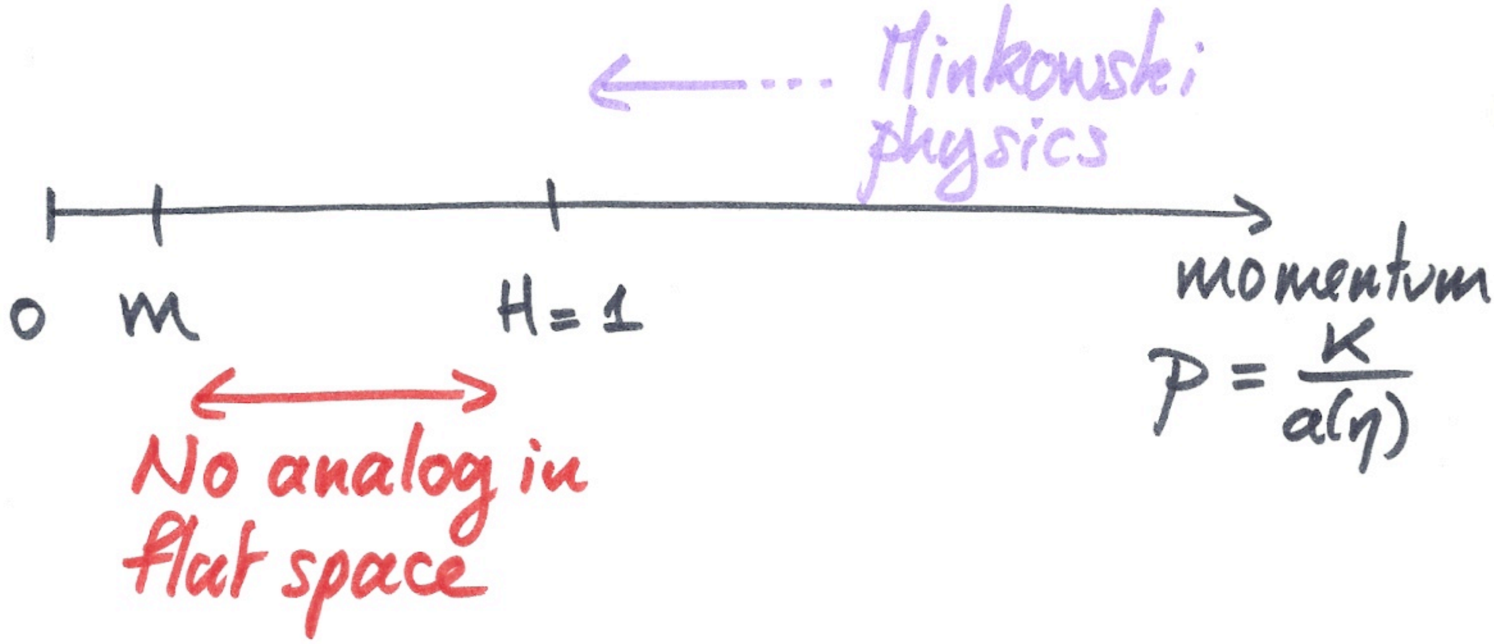
Redshift.



Stationary gravitational redshift leads to strong infrared (IR) fluctuations

Scalar fields in dS (III)

The case of light fields $m \ll H = 1$



Loop corrections :

 $\sim \dfrac{H^4}{m^2}$: IR divergencies

 $\sim H^2 \ln\left(P/H\right)$: large logs (secular divergencies)

$\Rightarrow$ $\boxed{\text{NEED FOR RESUMMATION}}$

Resummation / nonperturbative methods in de Sitter.

difficulty : nonequilibrium system

- Stochastic approach
  [Starobinsky, Yokoyama ('94)]
- Euclidean de Sitter
  [Rajaraman ('10); Beneke, Moch ('13)]
- Dynamical R.G.
  [Burgess et al. ('10)]
- Wigner-Weisskopf method
  [Boyanovsky ('12)]

- Large-N
  [Riotto, Sloth ('08); Mazzitelli, Paz ('89); J.S. ('11)]
- Dyson-Schwinger equations
  [Garbrecht, Rigopoulos ('11); Akhmedov, Burda ('12)
  Parentani, J.S. ('12); Gautier, J.S. ('13)]
- Nonperturbative / Functional R.G.
  [Kaya ('13); Serreau ('14); Guilleux, JS ('15)]

Resummation techniques in dS

Exploit full dS invariance

$$G(x,x') = \langle T\,\varphi(x)\varphi(x')\rangle = G(z)$$

dS invariant

$$(\Box + m^2)G(x,x') = \delta^{(D)}(x,x') + \underbrace{\int d^Dx''\,\Sigma(x,x'')\,G(x'',x')}$$

Difficult to formulate in terms of z only

[see, however, Youssef, Kreimer ('13)]

Exploit momentum space representation

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{X}.d\vec{X}\right], \quad D = d+1$$

$$\phi(x) = a(\eta)^{\frac{D-2}{2}}\varphi(x)$$

$$G_\phi(x,x') = G_\phi(\eta,\eta',|\vec{X}-\vec{X}'|) = \int\frac{d^dK}{(2\pi)^d}\,e^{i\vec{K}\cdot(\vec{X}-\vec{X}')}\,G_\phi(\eta,\eta',K)$$

$$\Rightarrow \left[\partial_\eta^2 + K^2 + m^2a^2 - \frac{a''}{a}\right]G_\phi(\eta,\eta',K) = \delta(\eta-\eta') + \int d\eta''\,\Sigma(\eta,\eta'',K)\,G(\eta'',\eta',K)$$

A typical nonequilibrium problem : Numerical solution ?

The p-representation

[Parentani, Serreau ('13) ; Adamek, Busch, Parentani ('13)]

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{x}.d\vec{x}\right]$$ ⇐ homogeneous non stationary

$$ds^2 = -(1-x^2)dt^2 - 2\vec{x}.d\vec{x}\,dt + d\vec{x}.d\vec{x}$$

⇐ inhomogeneous and stationary

$$G(\eta,\eta',K) = \frac{1}{K}\hat{G}(p,p')$$

$p = -K\eta$ and $p' = -K\eta'$: physical momenta

$$\Sigma(\eta,\eta',K) = K^3\,\hat{\Sigma}(p,p')$$

$$\left[\partial_p^2 + 1 - \frac{\nu^2 - 1/4}{p^2}\right]\hat{G}(p,p') = \delta(p-p') + \int dp''\,\hat{\Sigma}(p,p'')\,\hat{G}(p'',p')$$

$$\nu^2 = \frac{d^2}{4} - \frac{m^2}{H^2}$$

- Grav. redshift accounted for
- Reduces to a 0+1 dimensional problem

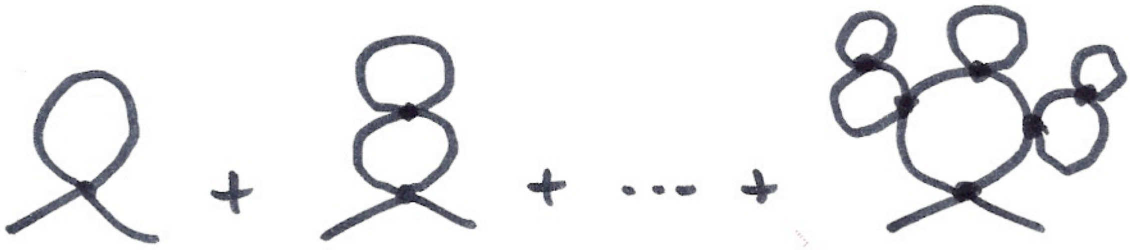
Application : O(N) theory

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4! N} (\varphi^2)^2 \right)$$

Large N

[Mazzitelli, Paz (89), Riotto, Sloth (08), Serreau (11)]

Local IR terms :



$\sim N$

Effective mass resumation

$$\sim \lambda \int \frac{d^dp}{(2\pi)^d} |H_\nu(p)|^2 \sim \frac{\lambda H^4}{M^2}$$

GAP EQUATION

$$M^2(\phi) = \underbrace{m^2 + \lambda\phi^2}_{m^2_{cl}(\phi)} + c\,\frac{\lambda H^4}{M^2(\phi)}$$

$$\boxed{M^2(\phi) = \frac{m^2_{cl}(\phi)}{2} + \sqrt{\frac{m^4_{cl}(\phi)}{4} + c\lambda H^4}}$$

Effective potential : symmetry restoration

[Serreau ('11)]

$$M^2(\phi^2) = \frac{m_{cl}^2(\phi^2)}{2} + \sqrt{\left(\frac{m_{cl}^2(\phi^2)}{2}\right)^2 + c\lambda H^4}$$

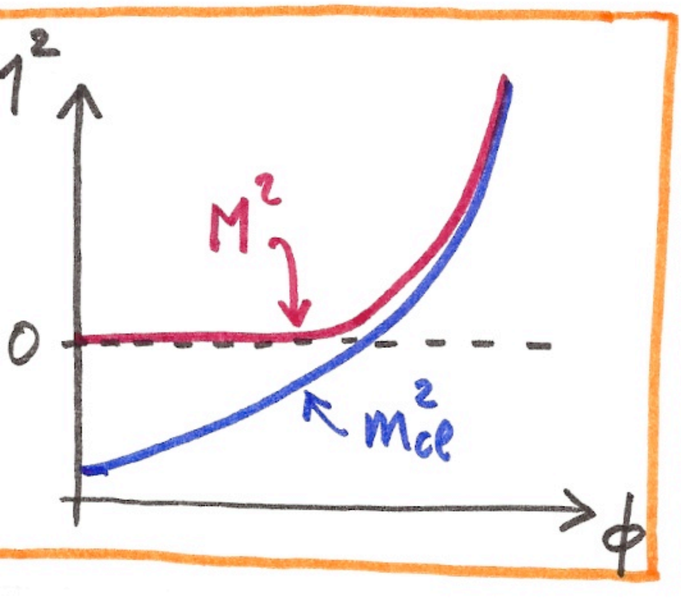
$$\boxed{V_{eff}(\phi) = \int_0^{\phi^2} du\, M^2(u)}$$

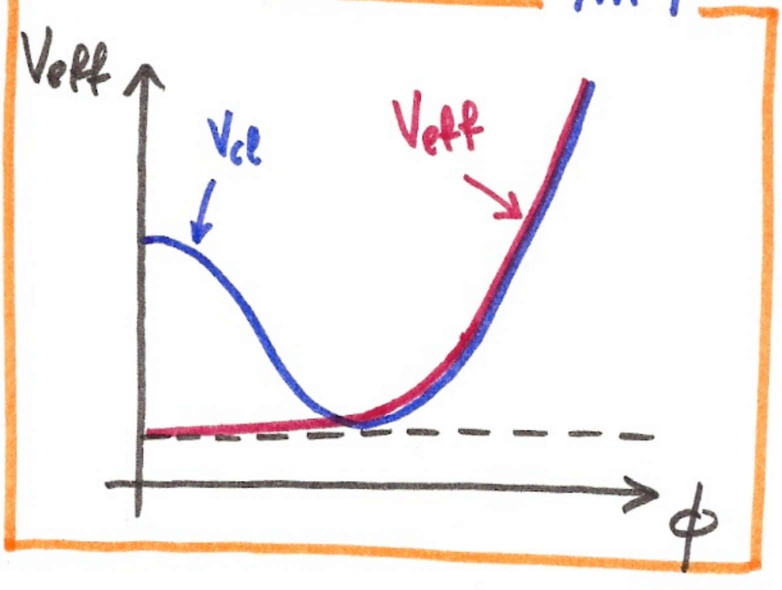
① $m_{cl}^2(\phi^2) > 0 \rightsquigarrow M^2(\phi^2) \approx m_{cl}^2(\phi^2)$

② $m_{cl}^2(\phi^2) = 0 \rightsquigarrow M^2(\phi^2) \approx \sqrt{c\lambda}\, H^2$

[Starobinsky, Yokoyama ('94)]

③ $m_{cl}^2(\phi^2) < 0 \rightsquigarrow M^2(\phi^2) \approx \dfrac{c\lambda H^4}{|m^2|}$



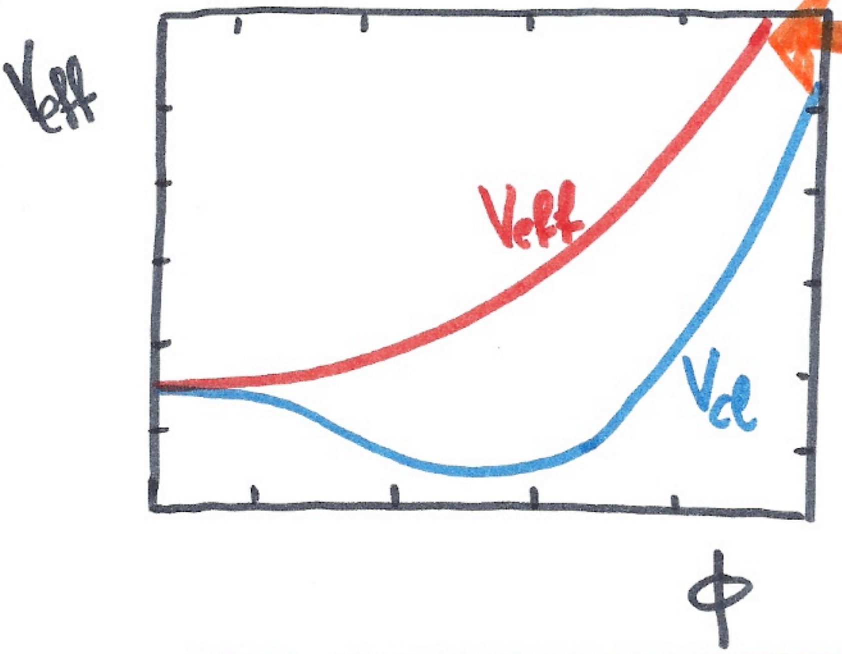


Large N : Effective potential

[Serreau, PRL (11)]

$$V_{eff}(\phi) = \int_0^{\phi^2} dx \, \frac{M^2(x)}{2}$$

$$= \frac{3}{2\lambda}\left(M^4(\phi) - M^4(0)\right) + \frac{3H^4}{16\pi^2} \ln \frac{M^2(\phi)}{M^2(0)}$$



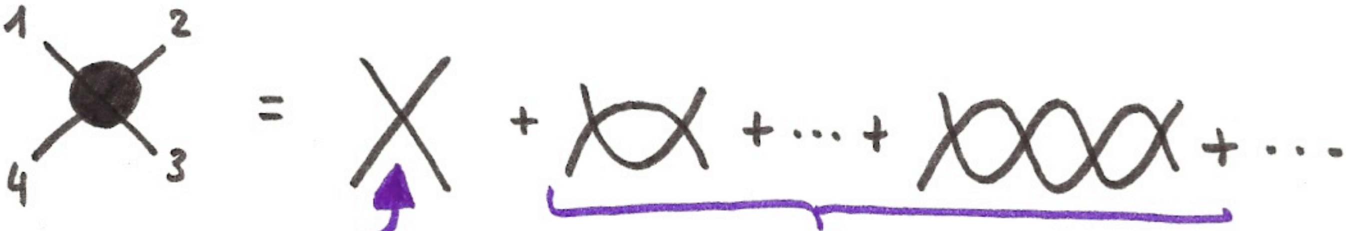
$\boxed{M^2(\phi) > 0}$

Radiative symmetry restoration
$\forall d$, $\forall H$

[See also : Ratra ('85); Mazzitelli et al. (89), Lazzari et al. (13)]

Four-point vertex : infrared logs.

[Serreau, Parentani ('13)]



$$\Gamma^{(4)}_{1234} = -\lambda\,\delta_{12}\delta_{13}\delta_{14} + i\,\delta_{12}\,\boxed{I_{13}}\,\delta_{34} + \text{perm.}$$

$$I(x,x') = \Pi(x,x') + i\int d^D z\, \Pi(x,z)\, I(z,x')$$

with $\Pi(x,x') = -\frac{\lambda}{6} G^2(x,x') =$ 

p-representation :

$$\hat{\Pi}(p,p') = -\frac{\lambda}{6}\int \frac{d^d q}{(2\pi)^d}\, \frac{\hat{G}(qp,qp')}{q}\, \frac{\hat{G}(rp,rp')}{r}$$

$r = |\vec{q}+\vec{e}\,|$, $\vec{e}$ unit vector

$$\hat{I}(p,p') = \hat{\Pi}(p,p') - i\int_C ds\, \hat{\Pi}(p,s)\, \hat{I}(s,p')$$

- Each bubble brings IR logs $\sim \ln(P/H)$
- The integral (SD) equation resums the infinite series of bubble diagrams

The IR dynamics is one-dimensional

IR modes $q p \lesssim H$ dominate the loop integral for IR external momenta $p, p' \lesssim H$

$$\nu = \sqrt{\frac{d^2}{4} - \frac{M^2}{H^2}} \equiv \frac{d}{2} - \varepsilon \quad ; \quad \varepsilon \ll 1$$

$$\boxed{\hat{\Pi}_F(p,p') = \frac{\Pi_F}{(pp')^{\kappa + 1/2}} \quad ; \quad \hat{\Pi}_\rho(p,p') = \frac{\Pi_\rho}{\sqrt{pp'}} \mathcal{P}_\nu^\varepsilon(\ln p/p')}$$

$$\kappa = \nu - \varepsilon \quad ; \quad \mathcal{P}_\nu^\varepsilon(x) = \frac{\operatorname{sh} \nu x}{\nu} e^{-\varepsilon |x|}$$

The bubble summation equation rewrites

$$\hat{I}_F(p,p') = \frac{\Pi_F H^{2\kappa}}{\sqrt{pp'}} \bar{A}\left(\ln \frac{p}{H}\right) \bar{A}\left(\ln \frac{p'}{H}\right)$$

$$\hat{I}_\rho(p,p') = \frac{\Pi_\rho}{\sqrt{pp'}} \bar{I}(\ln p/p')$$

$$\bar{I}(x) = \mathcal{P}_\nu^\varepsilon(x) + \Pi_\rho \int_x^0 dy \, \mathcal{P}_\nu^\varepsilon(x-y) \bar{I}(y)$$

$$\bar{A}(x) = e^{-\kappa x} + \Pi_\rho \int_x^0 dy \, \bar{I}(x-y) e^{-\kappa y}$$

One-dimensional integral equation

Exact Solution

$$P_\nu^\varepsilon(x) = \frac{sh(\nu x)}{\nu} e^{-\varepsilon |x|}$$

$$\bar{I}(x) = P_\nu^\varepsilon(x) + \pi_\rho \int_x^0 dy\, P_\nu^\varepsilon(x-y)\, \bar{I}(y)$$

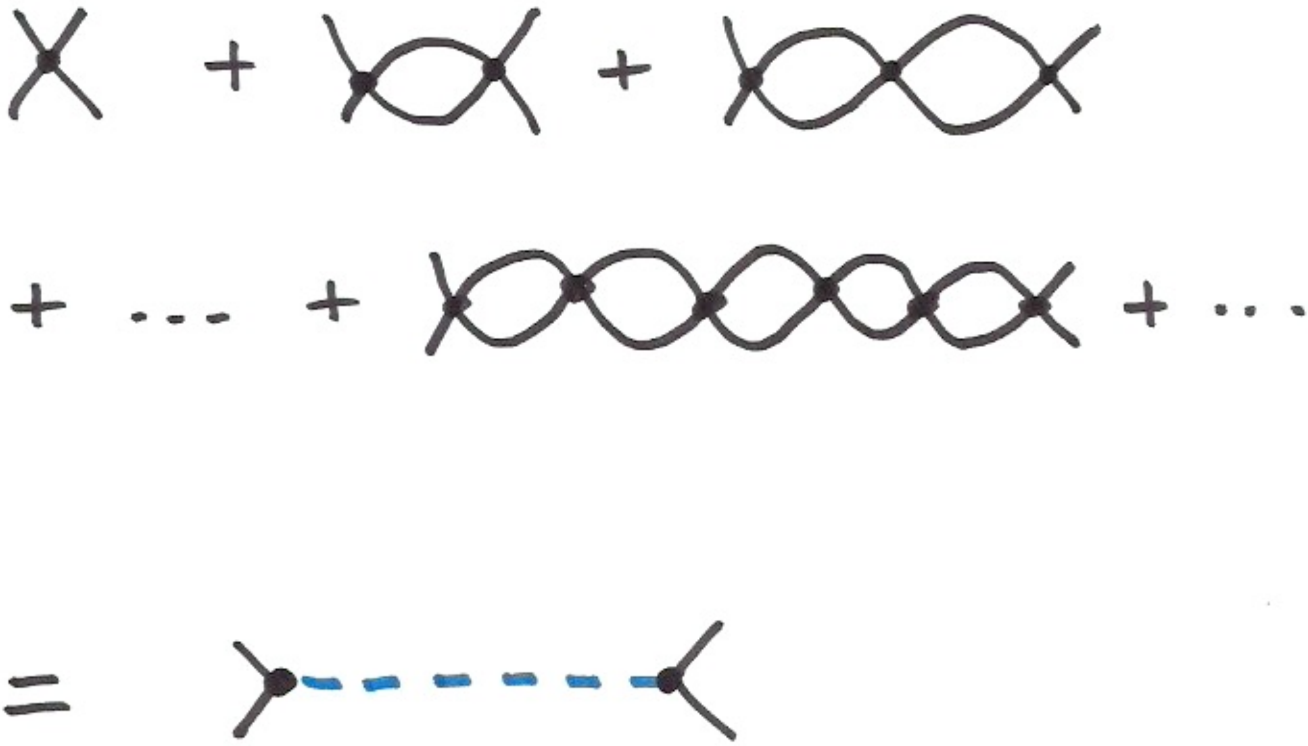
$$\boxed{\bar{I}(x) = P_{\bar{\nu}}^\varepsilon(x) \quad \text{with} \quad \bar{\nu} = \sqrt{\nu^2 + \pi_\rho}}$$

$$\hat{I}_F(p,p') = \frac{\pi_F}{(pp')^{\bar{\kappa}+1/2}} \; ; \; \bar{\kappa} = \bar{\nu} - \varepsilon$$

$$\hat{I}_\rho(p,p') = \frac{\pi_\rho}{\sqrt{pp'}} P_{\bar{\nu}}^\varepsilon(\ln p/p')$$

<u>The dangerous logs resum to a modified power law</u>

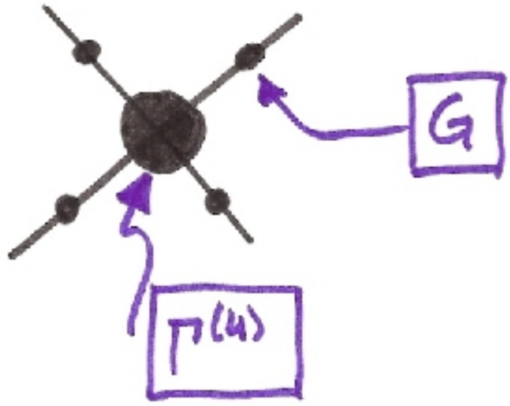
- Akin to anomalous dimension in critical phenomena
- Akin to mass correction in dS



Four-point correlator : non Gaussianities

[Serreau ('13)]

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle =$$



$$= \frac{\lambda}{3N} \frac{F_\nu^3}{2\nu} \frac{(-\eta)^{2-4\nu}}{(K_1 \dots K_4)^{2\nu}} \delta_{ab} \delta_{cd} \boxed{g\left(\ln \frac{\eta}{\eta_0}, \vec{K}_i\right)}$$

$$+ \text{perm.}$$

ex : massless case ($m_{dS}^2 = 0$) in the deep IR

$$g(x, \vec{K}_i) = -\frac{1}{4\varepsilon}\left(K_1^{2\nu} + \dots + K_4^{2\nu}\right) + \frac{1}{8\varepsilon} \frac{(K_1^{2\nu} + K_2^{2\nu})(K_3^{2\nu} + K_4^{2\nu})}{|\vec{K}_1 + \vec{K}_2|^{2\nu}}$$

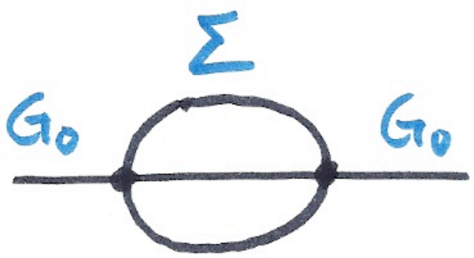
⇒ Non perturbative enhancement of loop contributions due to IR effects

⇒ Loop contrib's are of the same order as tree level ones.

Riotto, Sloth ('08)

Dyson-Schwinger Eqs. (DSE)

Nonlocal IR logarithms



$$\sim \quad \lambda^2 H^2 \ln(P/H)$$

→ need to resum the series

DSE : $$\boxed{G^{-1} = G_0^{-1} - \Sigma}$$

$$\boxed{(\Box + m^2)G(x,y) - \int_z \Sigma(x,z)G(z,y) = -i\delta(x-y)}$$

Integro-diff. eqn. with nontrivial metric

[Akhmedov et al. (12), (14); Gautier, J.S (13)
Garbrecht, Rigopoulos ('11)]

DSE : p - representation

$$G(x,x') = \int \frac{d^d K}{(2\pi)^d} e^{i\vec{K}.(\vec{x}-\vec{x}')} \tilde{G}(t,t',K)$$

$$\tilde{G}(t,t',K) = \frac{\hat{G}(p,p')}{K\,[a(t)a(t')]^{\frac{D-2}{2}}}$$

$$p = \frac{K}{a(t)} \quad , \quad p' = \frac{K}{a(t')}$$

Similarly : $\tilde{\Sigma}(t,t';K) = K^3 \dfrac{\hat{\Sigma}(p,p')}{[a(t)a(t')]^{\frac{D+2}{2}}}$

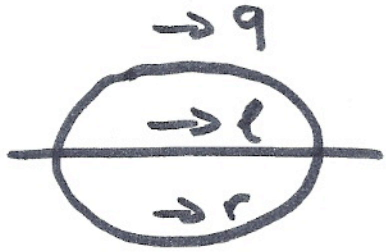
$$\left(\partial_p^2 + 1 - \frac{\nu^2 - 1/4}{p^2}\right)\hat{G}(p,p') + \int dp'' \hat{\Sigma}(p,p'')\hat{G}(p'',p') = i\delta(p-p')$$

$$\nu = \sqrt{\frac{d^2}{4} - \frac{M^2}{H^2}}$$

0+1 dimensional system

DSE : two-loop

[Gautier, J.S., PLB ('13)]



$$\underline{\hat{\Sigma}(p,p')} \propto \lambda^2 \int_{\vec{q},\vec{\ell}} \frac{\hat{G}_0(qp,qp')\,\hat{G}_0(\ell p,\ell p')\,\hat{G}_0(rp,rp')}{q\,\ell\,r}$$

$\boxed{\text{Exact analytical solution of DSE for } p,p' \ll 1}$

$$\hat{G}(p,p') = C_+\, G^{free}_{M_+}(p,p') + C_-\, G^{free}_{M_-}(p,p')$$

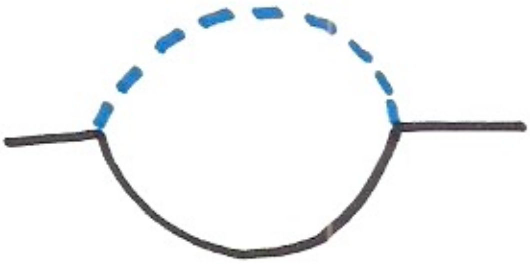
$$C_+ + C_- = 1$$

IR logs. resum to modified power laws + "splitting"

DSE : $1/N$ expansion

Σ^{NLO} =  + 

+ ⋯ +  + ⋯

=  $\sim \dfrac{1}{N}$

- Exact expression for IR momenta
- Exact solution of the DSE at NLO in $1/N$
  [Gautier, Serreau ('15)]

Two-point correlator

$$G(z) \propto \frac{C_+}{M_+^2\, z^{M_+^2/d}} + \frac{C_-}{M_-^2\, z^{M_-^2/d}}$$

Coincidence limit : $z = 1$

$$\langle \varphi^2(x) \rangle = G(1) \propto \frac{C_+}{M_+^2} + \frac{C_-}{M_-^2} \equiv \frac{1}{M_{dyn}^2}$$

Agrees with the Stochastic approach

Deep infrared (late time) limit

$$G(z) \sim \frac{C_+}{M_+^2} \, \frac{1}{z^{M_+^2/d}}$$

Amplitude

Power-law decay

Results at NLO in 1/N

$$\frac{m_{dyn}^2}{M_{LO}^2} = 1 + \frac{2}{N} \frac{\lambda_{eff}}{(1+\lambda_{eff})^2}$$

$$\frac{M_+^2}{M_{LO}^2} = 1 + \frac{1}{N} \frac{\lambda_{eff}(2-\lambda_{eff})}{(1+\lambda_{eff})^2}$$

$$\frac{M_+^2}{C_+ M_{LO}^2} = 1 + \frac{1}{2N} \frac{\lambda_{eff}(4+5\lambda_{eff})}{(1+\lambda_{eff})^3}$$

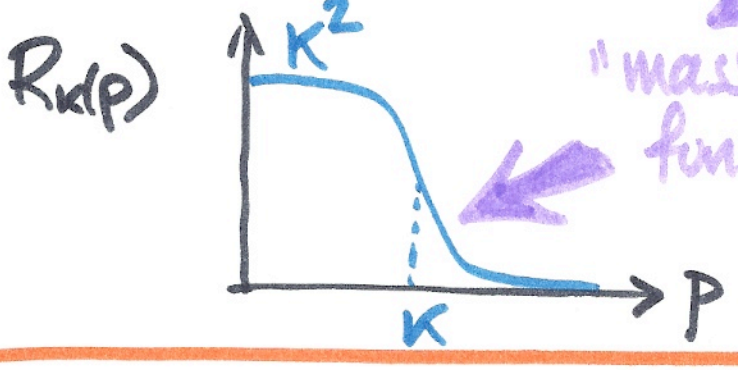
$$\lambda_{eff} = \frac{\lambda}{6\Omega_{D+1} M_{LO}^4}$$

Nonperturbative Renormalization Group (NPRG)

[Kaya ('13); Serreau ('14)]

- **Infrared regulator**:

$$S[\varphi] \longrightarrow S[\varphi] + \frac{1}{2}\int_{x,y} \varphi(x)\,R_\kappa(x,y)\,\varphi(y)$$

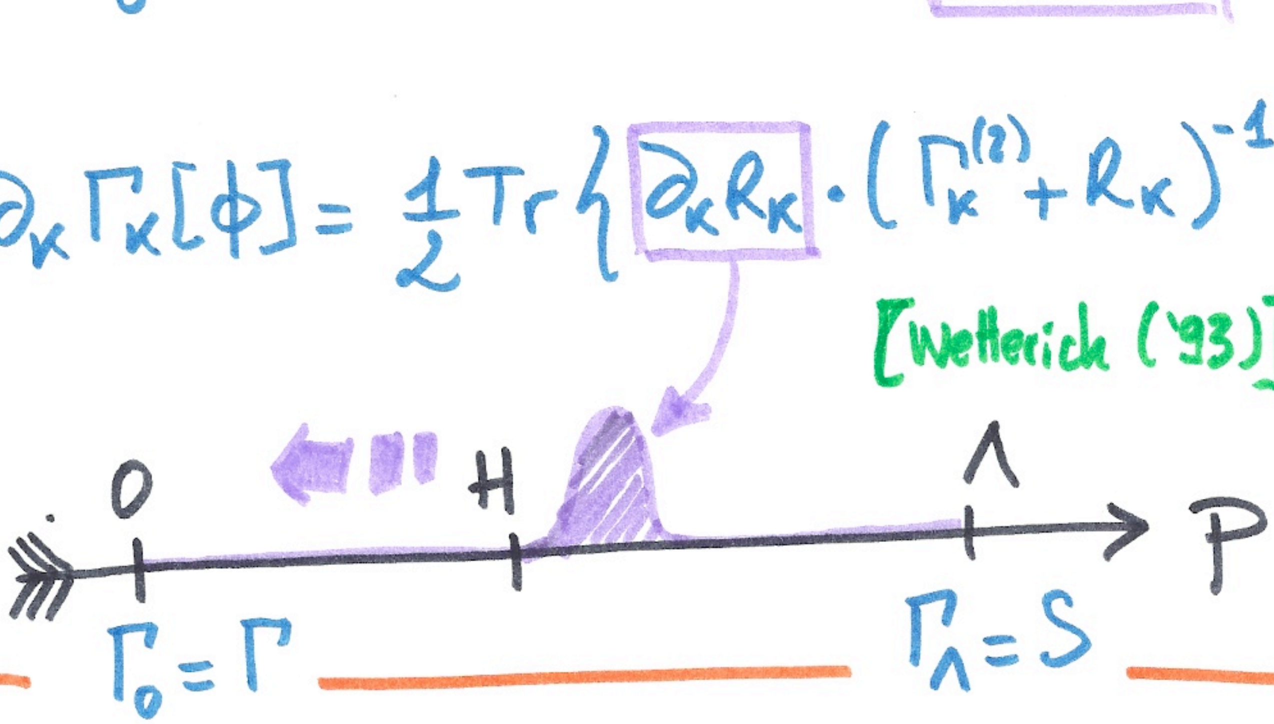


"mass function"

→ Regulated effective action $\boxed{\Gamma_\kappa[\phi]}$

$$\partial_\kappa \Gamma_\kappa[\phi] = \frac{1}{2}\,\mathrm{Tr}\left\{ \boxed{\partial_\kappa R_\kappa} \cdot \left(\Gamma_\kappa^{(2)} + R_\kappa\right)^{-1} \right\}$$

[Wetterich ('93)]



NPRG : Local Potential Approx. (LPA)

$$\Gamma_\kappa[\phi] = -\int_x \left\{ \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \boxed{V_\kappa(\phi)} \right\}$$

Full field dependence

+ choose $\quad R_\kappa(t,t';p) = \delta(t-t')(\kappa^2-p^2)\theta(\kappa^2-p^2)$

$$\boxed{\kappa\partial_\kappa V_\kappa = \frac{C_d\,\kappa^{d+2}}{\kappa^2+V_\kappa''}\,\underline{B_d}(\nu_\kappa,\kappa)}$$

$$C_d = \frac{\pi}{16d}\frac{\Omega_d}{(2\pi)^d} \quad , \quad \nu_\kappa = \sqrt{\frac{d^2}{4} - V_\kappa''}$$

$$\underline{B_d}(\nu,\kappa) = e^{-\pi\,\mathrm{Im}(\nu)}\Big\{(d^2-2\nu^2+2\kappa^2)\,|H_\nu(\kappa)|^2$$

$$+\,2\kappa^2|H_\nu'(\kappa)|^2 - 2d\kappa\,\mathrm{Re}\left[H_\nu^*(\kappa)H_\nu'(\kappa)\right]\Big\}$$

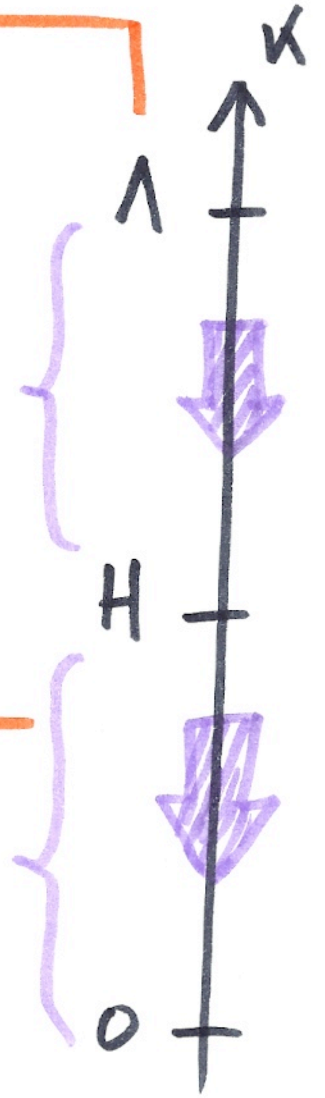
[Kaya ('13) Guilleux, Serreau ('15)]

From UV to IR : onset of gravitational effects

➡ $\kappa \gtrsim H$

$$\kappa \partial_\kappa V_\kappa \approx \frac{8 C_d}{\pi} \frac{\kappa^{d+2}}{\sqrt{\kappa^2 + V_\kappa''}}$$

Minkowski Regime



➡ $\kappa \lesssim H$ and $|V_\kappa''| \ll H^2$

$$\kappa \partial_\kappa V_\kappa \approx \frac{1}{\Omega_{D+1}} \frac{\kappa^2}{\kappa^2 + V_\kappa''}$$

[Compare to $\frac{\kappa^{D+2}}{\kappa^2 + V_\kappa''}$ in flat Euclidean space]

➡ **Effective dimensional reduction**

$D_{\text{eff}} = 0$

Zero-dimensional field theory

$$\boxed{e^{-\Omega_{D+1} W_\kappa(J)} = \int d\varphi \, e^{-\Omega_{D+1}\left(V_{eff}(\varphi) + J\varphi + \frac{\kappa^2}{2}\varphi^2\right)}}$$

$$V_\kappa(\phi) = W_\kappa(J) - J\phi - \frac{\kappa^2}{2}\phi^2$$

with $W'_\kappa(J) \equiv \phi$

$$\Rightarrow \boxed{\kappa \partial_\kappa V_\kappa(\phi) = \frac{1}{\Omega_{D+1}} \frac{\kappa^2}{\kappa^2 + V''_\kappa(\phi)}}$$

Adjust V_{eff} by matching at the scale $\kappa \approx H$

$$\boxed{V_{eff}(\varphi) \approx V_H(\varphi)}$$

Relation to the stochastic approach

[Starobinsky, Yokoyama ('94)]

Effective Langevin eqn. for IR modes

$$\dot{\varphi} + \frac{1}{d} V'_{soft}(\varphi) = \xi \quad ; \quad \langle \xi(t)\xi(t') \rangle = \frac{2\,\delta(t-t')}{d\,\Omega_{D+1}}$$

<u>Focker-Planck</u> <u>equation</u>

$$\partial_t P(\varphi,t) = \frac{1}{d}\frac{\partial}{\partial\varphi}\left\{ V'_{soft} P + \frac{1}{\Omega_{D+1}} P' \right\}$$

$$\boxed{P(\varphi, t \to \infty) \propto \exp\left\{ -\Omega_{D+1} V_{soft} \right\}}$$

$$\langle \varphi^2 \rangle = \int d\varphi\, P(\varphi)\, \varphi^2 = \frac{1}{\Omega_{D+1}} \underline{\underline{W''_{k=0}(J=0)}}$$

provided $V_{soft} = V_{eff} \approx V_H$

[see also Lazzari, Prokopec ('13)]

Relation to Euclidean de Sitter

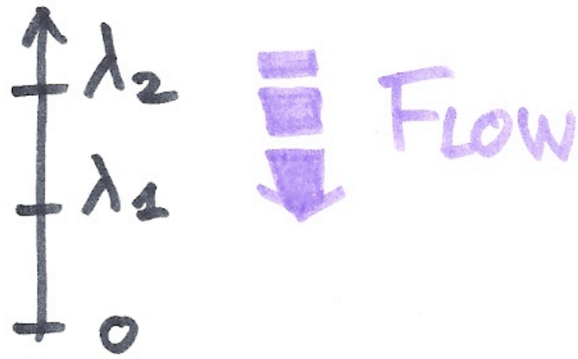
[Rajaraman ('10); Beneke, Moch ('13); Benedetti ('15)]

NPRG on S_D : $S \longrightarrow S + \frac{1}{2}\int_{xy} \varphi R_\kappa \varphi$

Compact space ⇒ discrete spectrum

$\Box Y_{\vec{L}} = -\lambda_L Y_{\vec{L}}$

$\lambda_L = L(L+D-1)H^2$



For $\boxed{\kappa^2 < \lambda_1}$, heavy modes decouple

⇒ zero mode only $\boxed{\bar{\varphi}}$

$$e^{-\Omega_{D+1}\bar{W}_\kappa(J)} = \int d\bar{\varphi}\, e^{-\Omega_{D+1}\left(\bar{V}_{eff}(\bar{\varphi}) + J\bar{\varphi} + \frac{\kappa^2}{2}\bar{\varphi}^2\right)}$$

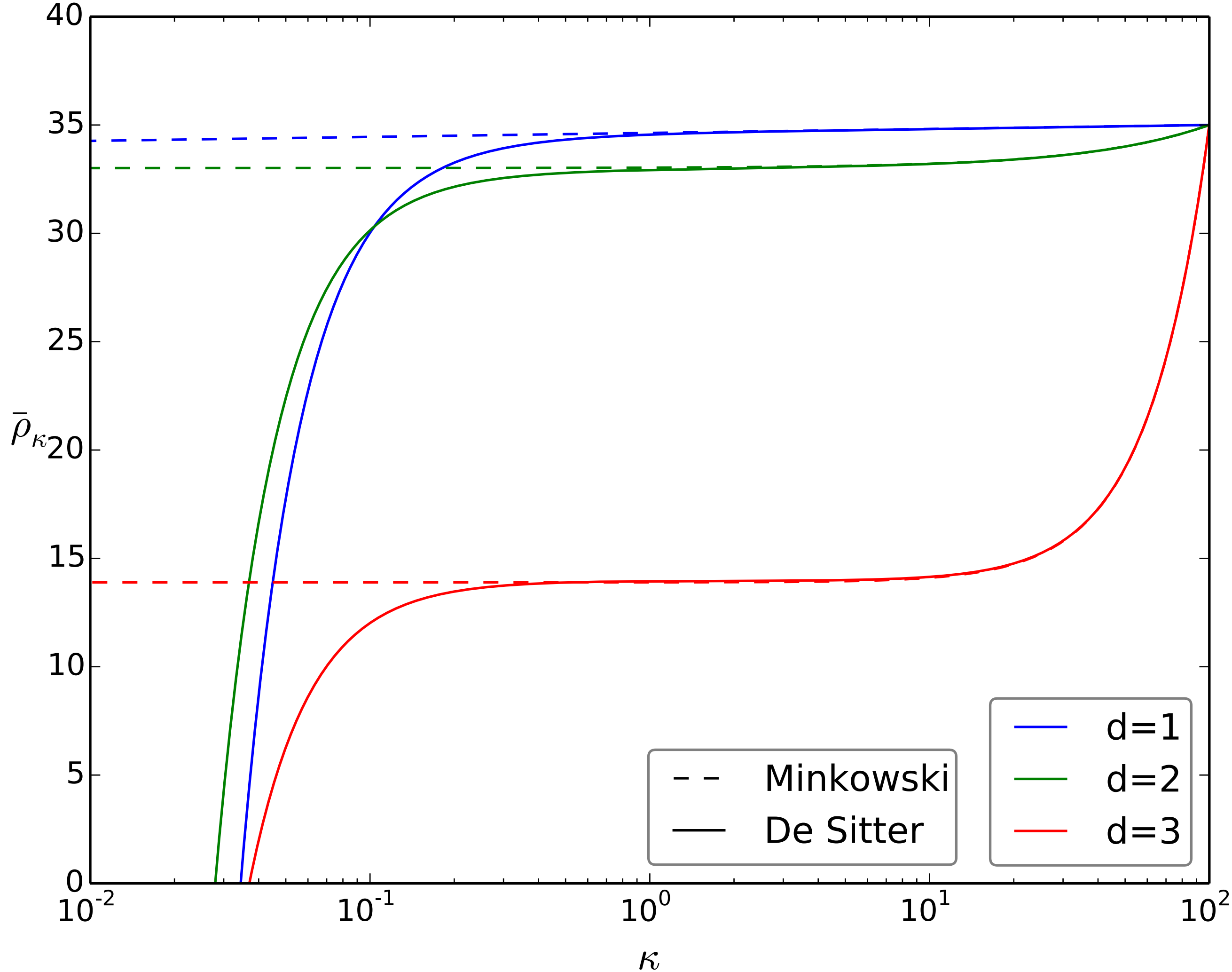
with $$e^{-\Omega_{D+1}\bar{V}_{eff}(\bar{\varphi})} = \int \mathcal{D}\hat{\varphi}\, e^{-\bar{S}[\bar{\varphi},\hat{\varphi}]}$$

From UV to IR: onset of gravitational effects

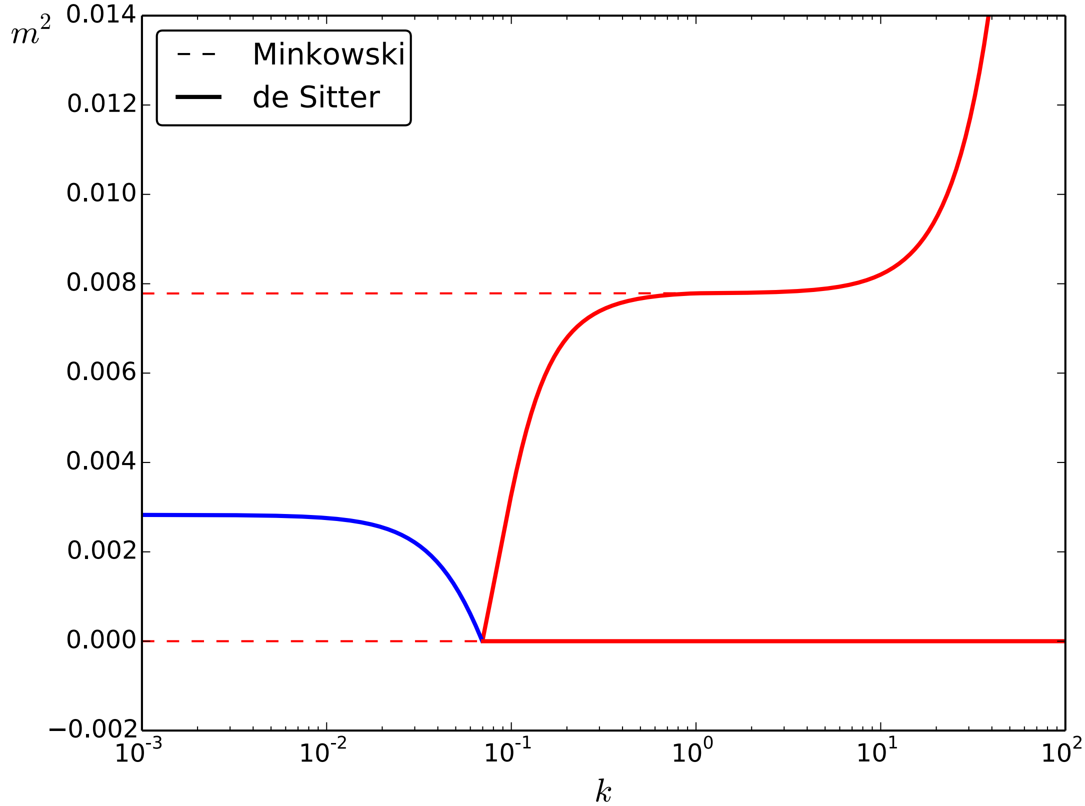
- Massive theory in UV ($\kappa \gtrsim H$)
(i.e. Minkowski symmetric phase)
$\Rightarrow V_H'' \gtrsim H^2$: Minkowski flow

- Massless modes / flat potential
(e.g. Minkowski critical theory or broken phase)
$\Rightarrow$ Regime of dimensional reduction

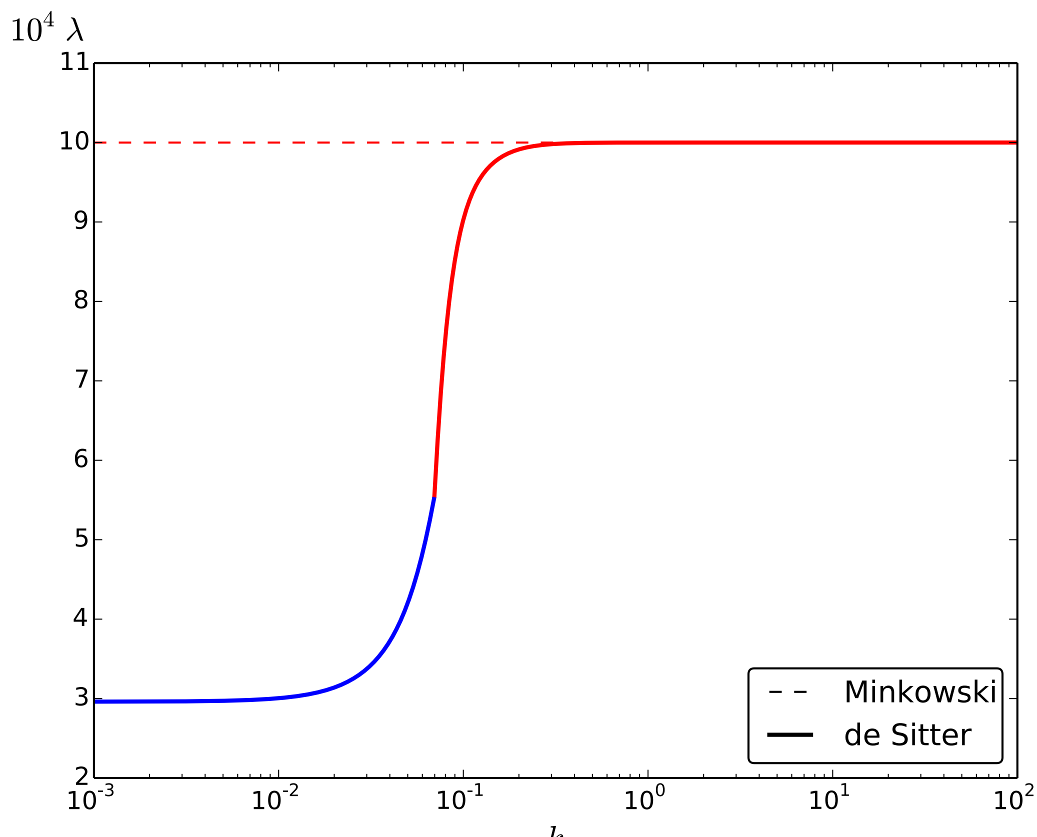
$\Rightarrow$ Symmetry restoration
Mass (re)generation

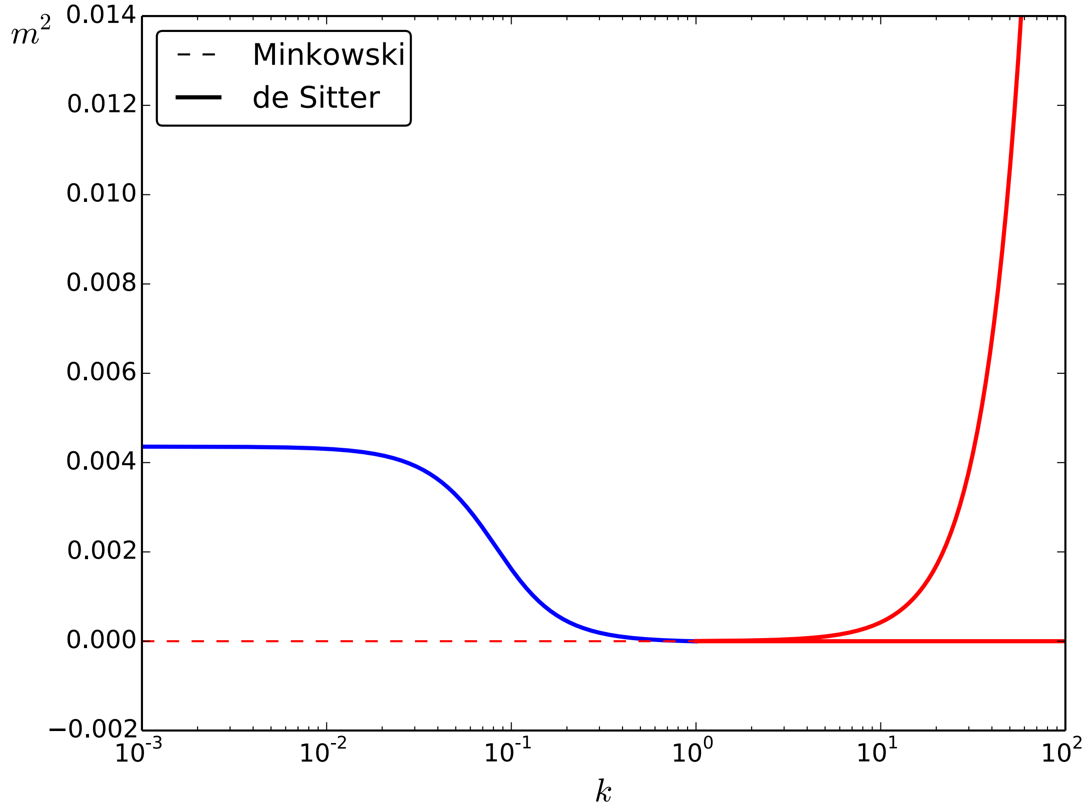


40
35
30
25
20
15
10
5
0
$\bar{\rho}_\kappa$
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
κ
Minkowski
De Sitter
d=1
d=2
d=3

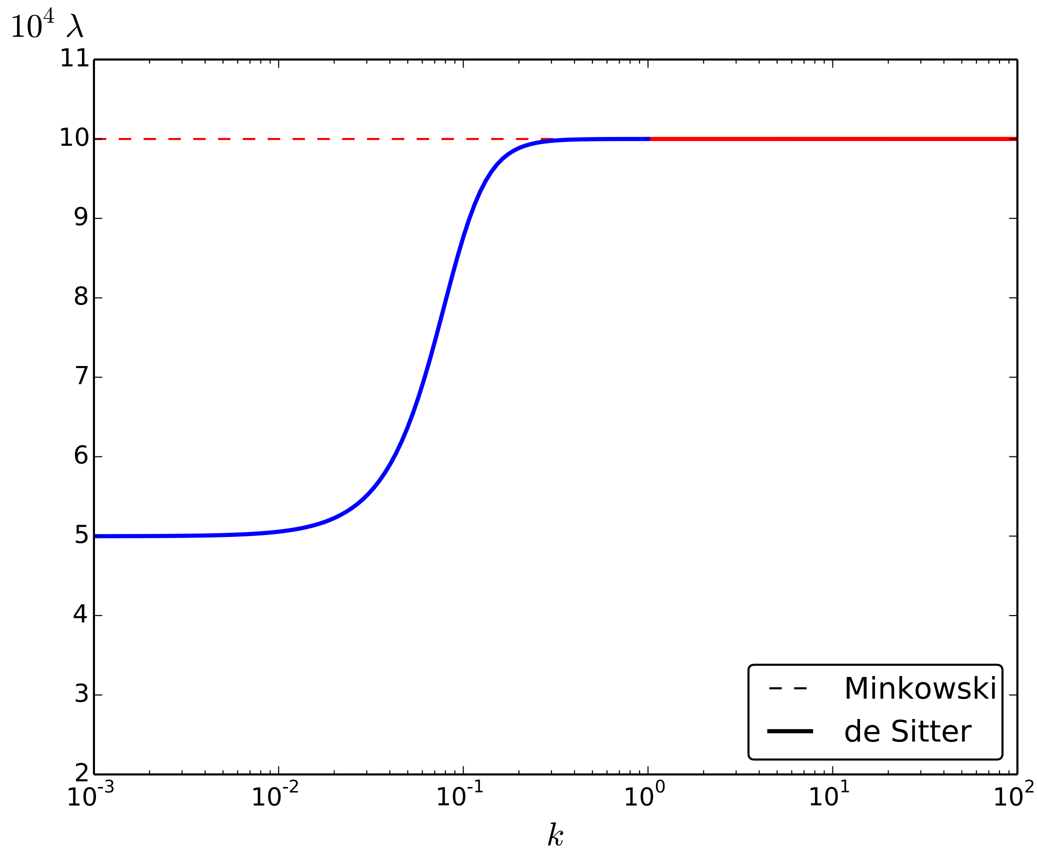


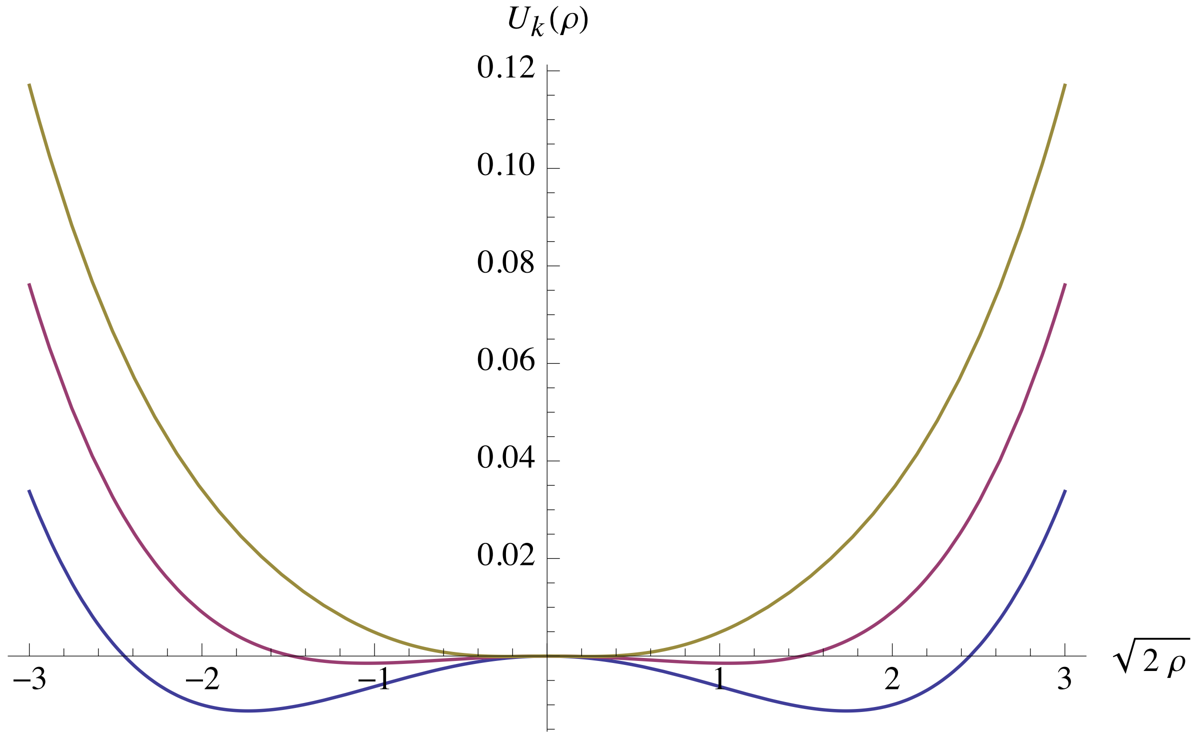
$N \to \infty$, "broken symmetry" case





$N \to \infty$, "critical" case





$N = 1$

OUTLOOK

- Rich, nonperturbative IR dynamics for light scalar fields in de Sitter space

- Nonperturbative / resummation techniques ⇒ nontrivial results

- ⇒ stochastic, large N, DSE, NPRG ...

- ⇒ mass generation, sym. restauration, O(1) loop corrections (non Gaussianities), nontrivial structure of correlators, dimensional reduction, relation between stochastic and RG flow ...