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Raman soliton generation from laser inputs in SRS

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Abstract

Stimulated Raman scattering of a laser pump pulse seeded by a Stokes pulse generically leaves a two-level medium initially at rest in an excited state constituted of *static solitons* and radiation. This is proved by building the IST solution of SRS on the semi-line, when the group velocity dispersion is not neglected. The procedure shows moreover that initial Stokes phase flips induce pump output with an extremely narrow spectral shape before complete depletion, related with Raman spikes of pump radiation. © 1999 Elsevier Science B.V.

1. Introduction

Stimulated Raman scattering (SRS) of high energy Laser pulse in two-level medium has been intensively studied these last years (see, e.g., the review [1]), more especially after the experiments by Drühl, Wenzel and Carlsten [2] showing that spikes of pump radiation (Raman spikes) occur spontaneously in the pump depletion zone, see also [3]. Consequently, nonlinear Raman amplification was used as a means to observe at the macroscopic level the fluctuation of the phase of the initial Stokes vacuum [4].

The SRS equations possess a Lax pair [5] and, as a consequence, theoretical and experimental works were partly devoted to the search of the *Raman soliton* predicted in [5]. In particular, it has been believed [6] that the *spike of pump radiation* observed in [2] is a soliton which, in the inverse spectral transform scheme (IST), would be related to the discrete part of the spectrum. However, it has been proved that the observed Raman spike is not a soliton but merely a manifestation of the continuous spectrum [7].

Then the question of the creation of a genuine Raman soliton is still an open problem (for 20 years now). We show here that the SRS equations, written with careful consideration of the *group velocity dispersion* and solved as a boundary value problem on the semi-line, do induce the generation of solitons by pair, and that, after the passage of the pulses, the solitons are static in the medium. Although the solitons are not seen in the output pump *intensity* (where only Raman spikes are seen), the signature of the soliton birth lies in the *phase* of the pump output, which remains to be experimentally measured.

The efficiency of the method of solution serves not only as a basis for seeking the Raman soliton, but also to evaluate *exactly* the result of any Raman amplification. Indeed, the SRS equations are solved for *arbitrary* values of the input ($x = 0$) pump and Stokes pulses on a medium with any initial ($t = 0$) state. The original problem is reduced to a Riccati equation (29) and linear integral equations (30). In many interest-

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ing cases the Riccati equation (29) can be explicitly solved and important physical information, such as the output laser fields, can be obtained without solving Eqs. (30).

As a consequence, we show that when the initial Stokes input is given a phase flip such as to generate a Raman spike [2,3], the pump output shows a long delay in depletion corresponding to a very narrow spectral shape. This process can be thought of as a means to generate highly monochromatic pulses.

2. The SRS equations

For completeness we provide hereafter the derivation of the SRS equations when group velocity dispersion (GVD) is taken into account. In such a situation it is necessary to consider the lineshapes of laser fields. We start with the standard model [9] for the interaction of light with a material medium schematically represented by a collection of harmonic oscillators (or *vibrons*) of frequency ω_V, mass m and amplitude of vibration $X\sqrt{2m}$. The molecular polarizability is expanded in Talor series as $\alpha + X\alpha'$ such that the macroscopic polarization is given in terms of the electric field of frequency ω by

$$\boldsymbol{P} = \epsilon_0 N[\alpha(\omega) + X\alpha']\boldsymbol{E}(\omega) . \tag{1}$$

The differential polarizability at equilibrium α' is assumed to be a constant, while the GVD is accounted for by the dependence of α on the light field frequency ω. In the following we shall consider polarized light beams and use only scalar fields.

The equation of motion for the dynamical variable X, together with the Maxwell equation for the electric field E, reads then [9]

$$X_{TT} + \omega_V^2 X = \epsilon_0\alpha' E^2 ,$$
$$c^2 E_{ZZ} - E_{TT} = N\alpha(\omega)E_{TT} + N\alpha'(XE)_{TT} , \tag{2}$$

which implies the following linear dispersion relation,

$$\omega = v(\omega)k , \quad v(\omega) = \left[\frac{c^2}{1 + N\alpha(\omega)}\right]^{1/2} . \tag{3}$$

The physical problem we are interested in is the interaction of two wave packets of peak frequencies ω_L and ω_S resonating on the medium transition frequency ω_V, that is to say obeying the Brillouin selection rules,

$$\omega_L - \omega_S = \omega_V , \quad k_L - k_S = k_V . \tag{4}$$

We shall make use of the notation $v_L = v(\omega_L)$, $v_S = v(\omega_S)$, and describe the slow variations of the envelopes of both waves by the multiscale expansion

$$E(Z,T) = \epsilon\, e^{i(k_L Z - \omega_L T)} \int d\omega\, a_L(\nu,\xi,\tau)\, e^{i(\kappa_L\xi - \nu\tau)}$$
$$+ \epsilon\, e^{i(k_S Z - \omega_S T)} \sqrt{\frac{\omega_S}{\omega_L}} \int d\omega\, a_S(\nu,\xi,\tau)\, e^{i(\kappa_S\xi - \nu\tau)}$$
$$+ \text{c.c.} , \tag{5}$$

in the slow variables

$$\xi = \epsilon Z , \quad \tau = \epsilon T . \tag{6}$$

Here above we have defined $\omega = \omega_L + \epsilon\nu$ (so as for $L \leftrightarrow S$), and hence the corresponding wave number variations are given by

$$\kappa_L(\nu) = \nu\,\frac{1 - \dot{v}_L k_L}{v_L} , \quad \dot{v}_L = \left.\frac{\partial v}{\partial\omega}\right|_{\omega_L} ,$$
$$L \leftrightarrow S . \tag{7}$$

The integration variable has been kept as ω to avoid a rescaling of the field amplitudes. Finally, the medium dynamical variable is also set in the form of a slowly varying envelope as

$$X(Z,T) = \epsilon Q(\xi,\tau)\, e^{i(k_V Z - \omega_V T)} + \text{c.c.} \tag{8}$$

Inserting these expressions in the system (2), and carefully considering the expansions

$$\alpha(\omega_L + \epsilon\nu) = \alpha_L + \epsilon\nu\,\dot{\alpha}_L , \quad \dot{\alpha}_L = \left.\frac{\partial\alpha}{\partial\omega}\right|_{\omega_L} ,$$
$$L \leftrightarrow S , \tag{9}$$

and the relation between $\dot{v}_L$ and $\dot{\alpha}_L$ resulting from (3), we finally arrive at

$$\left(\frac{\partial}{\partial\xi} + \frac{1}{v_L}\frac{\partial}{\partial\tau}\right) a_L(\nu)$$
$$= i v_L \beta Q a_S(\nu)\, e^{i[\kappa_S(\nu) - \kappa_L(\nu)]\xi} ,$$
$$\left(\frac{\partial}{\partial\xi} + \frac{1}{v_S}\frac{\partial}{\partial\tau}\right) a_S(\nu)$$
$$= i v_S \beta Q^* a_L(\nu)\, e^{-i[\kappa_S(\nu) - \kappa_L(\nu)]\xi} ,$$

$$-\mathrm{i}\omega_V \frac{\partial}{\partial\tau} Q = \epsilon_0 \alpha' \sqrt{\frac{\omega_S}{\omega_L}} \iint \mathrm{d}\omega\, \mathrm{d}\omega'\, a_L(\nu) a_S^*(\nu') \times \mathrm{e}^{\mathrm{i}[\kappa_L(\nu)-\kappa_S(\nu')]\xi}\, \mathrm{e}^{\mathrm{i}[\nu-\nu']\tau}, \tag{10}$$

where we have defined the constant

$$\beta = N\alpha' \frac{\sqrt{\omega_L \omega_S}}{2c^2}.$$

We seek now an integrable limit of the above system by assuming small group velocity dispersion, namely

$$v_S = v, \quad v_L = v + \epsilon\delta, \tag{11}$$

which implies a linear dependence on the frequency and hence

$$\dot v_L = \dot v_S = \frac{\epsilon\delta}{\omega_L - \omega_S}. \tag{12}$$

Consequently, the phase occurring in the system (10) becomes from (7) and $\omega_V/k_V = v + \mathcal{O}(\epsilon)$,

$$\kappa_S(\nu) - \kappa_L(\nu) = 2\epsilon\nu \frac{\delta}{v^2} = 2k, \tag{13}$$

which defines the *spectral parameter* k, proportional to ν, used now as the variable describing the spectral extension of input wave packets. Note that the zero group velocity dispersion limit $\delta \to 0$ corresponds to $k = 0$. Note also that $\mathrm{d}\omega = \epsilon\, \mathrm{d}\nu = v^2/\delta\, \mathrm{d}k$.

Defining finally a retarded frame as

$$x = \xi, \quad t = \tau - \frac{1}{v}\xi, \tag{14}$$

and the new variable

$$q(x,t) = \mathrm{i}v\beta Q(\xi,\tau), \tag{15}$$

the system of equations for the electric field components reads

$$\partial_x a_L = q a_S\, \mathrm{e}^{2\mathrm{i}kx}, \quad \partial_x a_S = -q^* a_L\, \mathrm{e}^{-2\mathrm{i}kx}. \tag{16}$$

In the evolution of the medium dynamical variable we keep a significant resonance term $\nu = \nu'$ and we arrive at

$$\partial_t q = -g \int \mathrm{d}k\, a_L a_S^*\, \mathrm{e}^{-2\mathrm{i}kx}, \tag{17}$$

where the coupling constant reads

$$g = \tfrac{1}{2}\epsilon_0 N(\alpha')^2 \frac{v}{\delta} \frac{v^2}{c^2} \frac{\omega_S}{\omega_V}. \tag{18}$$

The system (16), (17) can be solved on the infinite line $x \in (-\infty, +\infty)$ for arbitrary asymptotic boundary values [8] and it has been proved in [7] that this allows the interpretation of the experiments of [2]. We will demonstrate that it can be solved completely on the semi-line $x \in [0,\infty)$, which actually furnishes the solution at any point $x = L$ and hence solves the finite interval case as a *free end problem*. The input boundary values are defined as

$$a_L(k,0,t) = J_L(k,t), \quad a_S(k,0,t) = J_S(k,t), \tag{19}$$

where J_L and J_S are arbitrary.

3. Lax pair

The time evolution (17) results as the compatibility condition $U_t - V_x + [U,V] = 0$ for the Lax pair ($x \geq 0$, $t \geq 0$),

$$\varphi_x = U\varphi + \mathrm{i}k\varphi\sigma_3, \tag{20}$$

$$\varphi_t = V\varphi + \varphi\, \mathrm{e}^{-\mathrm{i}k\sigma_3 x}\, \Omega\, \mathrm{e}^{\mathrm{i}k\sigma_3 x}, \tag{21}$$

where the *dispersion relation* Ω is x-independent and

$$U = \begin{pmatrix} -\mathrm{i}k & q \\ -\bar q & \mathrm{i}k \end{pmatrix},$$

$$V = \frac{g}{4\mathrm{i}} \int \frac{\mathrm{d}\lambda}{\lambda - k} \begin{pmatrix} |a_L|^2 - |a_S|^2 & 2a_L \bar a_S\, \mathrm{e}^{-2\mathrm{i}\lambda x} \\ 2\bar a_L a_S\, \mathrm{e}^{2\mathrm{i}\lambda x} & |a_S|^2 - |a_L|^2 \end{pmatrix}. \tag{22}$$

$V(k)$ is discontinuous, therefore only the limits $V^\pm = V(k \pm \mathrm{i}0)$ are defined on the real k-axis. Such is also the case for Ω, and $\Omega^\pm$ will have to be determined from the boundary conditions on the solution φ.

4. Scattering problem

Following standard methods in spectral theory, see e.g. [10], we define the solutions φ^+ and φ^- according to

$$\begin{pmatrix} \varphi_{11}^+ \\ \varphi_{21}^+ \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\int_0^x \mathrm{d}\xi\, q\varphi_{21}^+ \\ \int_0^x \mathrm{d}\xi\, \bar q \varphi_{11}^+\, \mathrm{e}^{2\mathrm{i}k(x-\xi)} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{12}^+ \\ \varphi_{22}^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \int_x^\infty d\xi\, q\varphi_{22}^+ \, e^{-2ik(x-\xi)} \\ \int_0^x d\xi\, \bar{q}\varphi_{12}^+ \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{11}^- \\ \varphi_{21}^- \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^x d\xi\, q\varphi_{21}^- \\ \int_x^\infty d\xi\, \bar{q}\varphi_{11}^- \, e^{2ik(x-\xi)} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{12}^- \\ \varphi_{22}^- \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^x d\xi\, q\varphi_{22}^- \, e^{-2ik(x-\xi)} \\ -\int_0^x d\xi\, \bar{q}\varphi_{12}^- \end{pmatrix}. \tag{23}$$

These solutions obey the *reduction* $\bar{\varphi}_1^+(\bar{k}) = i\sigma_2 \times \varphi_2^-(k)$, the Riemann–Hilbert relations

$$\varphi_1^- = \varphi_1^+ - e^{2ikx} \rho^* \varphi_2^- ,$$
$$\varphi_2^+ = \varphi_2^- + e^{-2ikx} \rho\varphi_1^+ , \tag{24}$$

and the bounds

$$x = 0 : \quad \varphi^+ = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}, \quad \varphi^- = \begin{pmatrix} 1 & 0 \\ -\rho^* & 1 \end{pmatrix},$$
$$x \to \infty : \quad \varphi^+ \to \begin{pmatrix} 1/\tau & 0 \\ e^{2ikx}\rho^*/\tau^* & \tau \end{pmatrix},$$
$$\varphi^- \to \begin{pmatrix} \tau^* & -e^{-2ikx}\rho/\tau \\ 0 & 1/\tau^* \end{pmatrix}, \tag{25}$$

which define the reflection coefficient ρ and the transmission coefficient τ ($\rho^*(k)$ stands for $\bar{\rho}(\bar{k})$). Note that $\det\{\varphi(x)\} = \det\{\varphi(\infty)\} = 1$, hence for real k,

$$|\tau|^2 = 1 + |\rho|^2 . \tag{26}$$

The column vectors φ_1^+ and φ_2^- are entire functions in the k-plane, with good behaviors as $k \to \infty$ in the upper half-plane for φ_1^+ (and the lower one for φ_2^-). The vector φ_2^+ is meromorphic in the upper half-plane with a finite number N of simple poles k_n (and φ_1^- in the lower one with poles $\bar{k}_n$). Consequently, τ and ρ have meromorphic extensions in the upper half-plane where they possess the N simple poles k_n (the bound states locations). In particular, we have the relation obtained by a limit procedure on (23),

$$\operatorname*{Res}_{k_n}\{\varphi_2^+\} = c_n^+ \varphi_1^+(k_n^+) \exp[-2ik_n^+ x] ,$$
$$c_n^{\pm} = \operatorname*{Res}_{k_n} \rho . \tag{27}$$

5. Solution of SRS

From the boundary data (19) we have readily

$$\begin{pmatrix} a_L \\ a_S\, e^{2ikx} \end{pmatrix} = J_L \varphi_1^+ + J_S \varphi_2^- \, e^{2ikx} . \tag{28}$$

After careful computations of the boundary values of (21), using separately φ^+ and φ^-, we obtain both limits Ω^+ and Ω^- of Ω and the following time evolution of the scattering coefficient $\rho(k,t)$,

$$\rho_t = -\rho^2 C^+[m^*] - 2\rho C^+[\phi] - C^+[m] ,$$
$$C^+[f] = \frac{1}{\pi} \int \frac{d\lambda}{\lambda - (k + i0)} f(\lambda) ,$$
$$m = \frac{i\pi}{2} g J_L J_S^* , \quad \phi = \frac{i\pi}{4} g(|J_L|^2 - |J_S|^2) . \tag{29}$$

The reconstruction proceeds through the solution of the Riemann–Hilbert problem (24), which reads

$$\varphi_1^+(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2i\pi} \int_{C_-} \frac{d\lambda}{\lambda - k} \rho^*(\lambda) \varphi_2^-(\lambda)\, e^{2i\lambda x} ,$$
$$\varphi_2^-(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi} \int_{C_+} \frac{d\lambda}{\lambda - k} \rho(\lambda) \varphi_1^+(\lambda)\, e^{-2i\lambda x} , \tag{30}$$

where C_+ (resp. C_-) is a contour from $-\infty + i0$ to $+\infty + i0$ (resp. from $-\infty - i0$ to $+\infty - i0$) passing over all poles k_n of $\rho(k)$ (resp. under $\bar{k}_n$). Remember the notation $\rho^*(\lambda) = \bar{\rho}(\bar{\lambda})$.

Since m and ϕ are continuous bounded functions of k, the coefficients of the Riccati equation (29) are analytic functions in the upper half k-plane, where precisely ρ is meromorphic at time $t = 0$. Consequently, ρ remains meromorphic at later times, which allows to prove (see appendix) that the reconstructed fields do obey the system of equations (16), (17), or in short that the reconstructed φ does obey (25). Without such a theorem, the method would fail.

The formulae (29) and (30) with (28) furnish the solution of (16), (17), in particular the output field values at $x = L$ (whatever may be q for $x > L$). In the limit $L \to \infty$, and by (25), the solution becomes *explicit*, and for instance the output pump reads

$$x \to \infty : \quad a_L \to \frac{1}{\tau} J_L - \frac{\rho}{\tau} J_S . \tag{31}$$

6. Evolution of laser pulses

We are interested here in the physically relevant case when the medium is initially in the ground state, that is $\rho(k,0) = 0$. Next we choose a Stokes wave as a portion of the pump wave, both with Lorentzian lineshape, namely, for $\kappa > 0$,

$$|J_L|^2 = |A(t)|^2 \frac{\kappa}{\pi(k^2+\kappa^2)}, \quad J_S = e^{-\gamma - i\theta(t)} J_L,$$

where $A(t)$ is the pulse shape of duration t_m ($t > t_m \Rightarrow A(t) = 0$). The parameter γ (real positive constant) measures the ratio pump/Stokes inputs. Considering the case when the input Stokes wave experiences one phase flip at some time t_0, we choose the phase $\theta(t) = 0$ for $t < t_0$ and $\theta(t) = \pi$ for $t > t_0$.

It follows from contour integration that the Riccati equation (29) can be rewritten as

$$\rho_t = -\frac{i}{2}\frac{g e^{-\gamma}}{k + i\kappa}|A|^2 \left[\rho^2 e^{-i\theta} - 2\rho \sinh\gamma - e^{i\theta}\right].$$

Taking into account the phase flip, the general solution with zero initial datum reads

$$t \le t_0: \quad \rho = \frac{\sinh\delta}{\cosh(\delta - \gamma)}, \tag{32}$$

$$t \ge t_0: \quad \rho = \frac{\rho_0 \cosh(\delta - \delta_0 + \gamma) - \sinh(\delta - \delta_0)}{\cosh(\delta - \delta_0 - \gamma) - \rho_0 \sinh(\delta - \delta_0)}, \tag{33}$$

with $\rho_0 = \rho(k, t_0)$, $\delta_0 = \delta(k, t_0)$ and the definitions

$$\delta(k,t) = \frac{iT(t)}{k + i\kappa},$$

$$T(t) = \tfrac{1}{4} g(1 + e^{-2\gamma}) \int_0^t dt' \, |A|^2. \tag{34}$$

7. Soliton generation

Let us consider first the case with no phase flip, that is $t_0 > t_m$. The spectral transform ρ, given by (32), has an infinite set of moving poles $k_n(t)$, $n \in \mathbb{Z}$, given by

$$k_n = -i\kappa + \frac{T(t)}{(n + \frac{1}{2})\pi - i\gamma}, \tag{35}$$

which are associated with solitons as soon as they lie in the upper half plane (in the lower half plane they are the resonances).

As t evolves, and for a given linewidth κ, these poles move from the point $-i\kappa$ and they may cross the real axis (and generate solitons) if $T(t)$ is large enough, which means enough energy in the input pulses. Moreover, since $k_n = -\bar{k}_{-n-1}$, solitons are created by pair. After the passage of the pulse ($t > t_m$), m and ϕ vanish in (29), $T(t)$ is constant, and hence the whole solution becomes t-independent. Consequently, the laser pulses leave in the medium a *finite number of static bi-solitons.*

The question of their observation is not straightforward. Indeed, taking for simplicity the case of an infinite medium, the pump output is given in (31) where the coefficients ρ and τ are understood for real values of k (the essential mismatch). The main point here is that both $1/\tau$ and ρ/τ are holomorphic functions in the upper half-plane of k, and hence an integrated intensity like $\int dk\,|a_L|^2$ will not show the presence of the poles k_n. Moreover, if at $t = t_s$ a resonance crosses the real axis at k_s to become a bound state (soliton birth), the output *intensity* at that particular value k_s will simply show a full depletion ($1/\tau(k_s) = 0$ and $|\rho(k_s)/\tau(k_s)| = 1$).

We remark here that, in the *sharp line limit* $\kappa \to 0$, $g\kappa \to$ *cste*, and without phase flip in the Stokes input, the infinite set of poles k_n coalesce in $k = 0$ which implies that the solution becomes self-similar (the self-similar solution of SRS is analyzed in [12]). A self similar solution of SRS in the case of zero GVD (and without Stokes phase flip) was recently and independently demonstrated in [11] by a different technique.

8. Stokes phase flip and Raman spike

As shown in [7], the *spike of pump radiation* observed in [2] and largely studied later, occurs at time t_r for which $\rho(k, t_r) = 0$. Indeed, then $|\tau(k, t_r)| = 1$ and from (31) the pump (intensity) is fully repleted. The spike (in the time domain) can then be seen only for long pulses, for which damping (homogeneous broadening) allows the spike to take place in the depletion region of the pump output. In our case, however, as damping is neglected, the spike is not visible

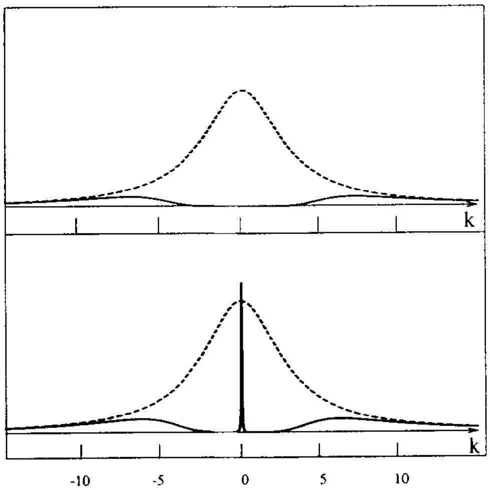

Fig. 1. Pump output intensities (arbitrary units) at $T = 0$ (dashed line) and $T = 10$ (full line) with no phase flip for the upper figure and a phase flip at $T = 1.257841$ for the lower one.

in the time domain but observable in the spectral domain as shown below.

The expression in (33) gives that ρ vanishes at t_r for

$$\coth(\delta_0) - \coth(\delta_r - \delta_0) = 2\tanh\gamma , \tag{36}$$

which relates t_0 (instant of phase flip) and t_r (instant of Raman spike) to the pulse area through (34). We plot in Fig. 1 the output intensities $|a_L(k)|^2$ as given by (31), for different values of the normalized time $T(t)$ defined in (34), namely $T = 0$ (dashed lines) and $T = 10$ (full lines). In the absence of phase flip in the initial Stokes input, the effect of pump depletion (Stokes amplification) is seen on the upper figure where all frequencies components (with enough energy) are depleted. However, when a phase flip is introduced at time t_0 such that Eq. (36) possess a solution, the $k = 0$ mode, corresponding to the very frequency ω_L, is not depleted for a long time (here the spike starts to decrease after $T = 28$). The parameter chosen to illustrate the spike are $\kappa = 3$, $\gamma = 1$ and a normalized input $\int dk\, |J_L(k,t)|^2 = 1$, which actually means that $|A(t)|^2 = 1$ (the field amplitudes can be normalized to arbitrary units).

We note finally that the occurrence of a phase flip alters the denominator of ρ, which can be used to change the discrete spectrum and hence to modify the soliton part of the solution.

9. Conclusion

The following results have been obtained:

(1) The proof that genuine Raman solitons do occur in SRS experiments and cannot be detected in the energy of the output but probably in its phase.

(2) The *spontaneous* generation of solitons out of vacuum in an integrable system.

(3) The complete solution by IST of the boundary value problem for SRS on the semi-line. This solution relies on the analytical properties of the nonlinear Fourier transform $\rho(k,t)$ which are *conserved* in the time evolution.

(4) The proof that Raman spikes are *generic* as they occur as well for short duration pulses, as suspected in the experimental work [3].

(5) The proof that in the sharp line limit and with the proportional pump/Stokes input on a medium initially at rest, the solution is self-similar.

(6) The striking spectral distribution of the output laser pump in the presence of initial Stokes phase flip.

10. Comments

(1) The IST solution of SRS on the semi-line has been considered in [13] for the sharp line case by using an approximate evolution equation for the reflection coefficient.

(2) IST on the semi-line for the nonlinear Schrödinger equation has been considered in [14], and the problem is considerably more difficult mainly because of the necessity of the knowledge of additional constraints in $x = 0$ (e.g., the time derivative of the field). The initial-boundary value problem for SRS in the case of exact resonance where recently studied in [11]. Initial problems with asymptotic boundary condition for Self Induced Transparency equations were solved in [8]. The initial boundary value problems for SIT equations were tackled independently by one of the authors (A.V.M., in 1986, unpublished) and by [15].

(3) At the moment we are not able to extract, from the explicit form of the spectral data, the explicit form of the *potential* (the medium excitation field), mainly because ρ has an essential

singularity in $k = i\kappa$. However, the method provides the explicit form of the physically relevant quantities which are the light pulses output values. We believe that the theory of approximate solutions of the inverse scattering problem can be developed.

Appendix A

We prove here the following.

Theorem 1. the solution $\phi(k,x,t)$ of the Cauchy-Green integral equation

$$\phi_1(k) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2i\pi}\int_{C_-} \frac{d\lambda}{\lambda - k}\rho^*(\lambda)\phi_2^-(\lambda)\, e^{2i\lambda x},$$

$$\phi_2(k) = \begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2i\pi}\int_{C_+} \frac{d\lambda}{\lambda - k}\rho(\lambda)\phi_1^+(\lambda)\, e^{-2i\lambda x}, \quad \text{(A.1)}$$

coincides with the solution $\varphi(k,x,t)$ of (23) if

(1) the reflection coefficient ρ possess a meromorphic continuation in the upper half-plane with simple poles k_n and residues c_n, and

(2) the *potential* is given from ϕ by

$$q(x,t) = 2i\phi_{12}^{(1)}(x,t), \quad \text{(A.2)}$$

where $\phi_{12}^{(1)}$ denotes the coefficient of $1/k$ in the Laurent expansion of the matrix element $\phi_{12}^{(1)}(k,x,t)$.

Proof. The first step is to verify from (24) that the function ϕ is solution of the differential problem (20). This is easily done by following the method of [17], in short, prove the relation

$$\frac{\partial}{\partial \bar{k}}[(\phi_x - ik\phi\sigma_3)\phi^{-1}] = 0, \quad \text{(A.3)}$$

and integrate it with the information

$$\phi(k) = \mathbf{1} + \frac{1}{k}\phi^{(1)} + \cdots. \quad \text{(A.4)}$$

Then the solution ϕ of (30) solves the spectral problem (20) with the potential (A.2).

The second step consists in proving

$$\phi_1^+ = \varphi_1^+, \quad \phi_2^- = \varphi_2^-, \quad \text{(A.5)}$$

which is achieved just by comparing the values of these vectors in $x = 0$. Indeed from (A.1), the Cauchy theorem leads to

$$\forall x < 0: \quad \phi_1^+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \phi_2^- = \begin{pmatrix}0\\1\end{pmatrix}. \quad \text{(A.6)}$$

Consequently, the two matrices (ϕ_1^+, ϕ_2^-) and $(\varphi_1^+, \varphi_2^-)$ solve the same first order differential problem and have the same values in $x = 0$, so (A.5) is proved.

The last step results in the proof that

$$\phi_1^- = \varphi_1^-, \quad \phi_2^+ = \varphi_2^+, \quad \text{(A.7)}$$

which has to be done in the same way but with the behaviors as $x \to \infty$ instead of the bounds in $x = 0$. As $x \to \infty$, the behavior of ϕ is not well defined and it is convenient to work with the new function

$$\mu(k,x,t) = \exp[ik\sigma_3 x]\phi(k,x,t)\exp[-ik\sigma_3 x], \quad \text{(A.8)}$$

which from (A.1) verifies

$$\mu_1^- = \mu_1^+ - \bar{\rho}\mu_2^-,$$
$$\mu_2^+ = \mu_2^- + \rho\mu_1^+, \quad \text{(A.9)}$$

From (A.5) and (25) we already have as $x \to \infty$,

$$\mu_1^+ \to \begin{pmatrix}1/\tau\\ \bar{\rho}/\bar{\tau}\end{pmatrix}, \quad \mu_2^- \to \begin{pmatrix}-\rho/\tau\\ 1/\bar{\tau}\end{pmatrix}, \quad \text{(A.10)}$$

hence as $x \to \infty$, we get by means of (26),

$$\mu_1^- \to \begin{pmatrix}\bar{\tau}\\0\end{pmatrix}, \quad \mu_2^+ \to \begin{pmatrix}0\\ \tau\end{pmatrix}, \quad \text{(A.11)}$$

which proves (A.7) and ends the demonstration of the theorem. □

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References

[1] M.G. Raymer, I.A. Walmsley, Prog. Opt. 28 (1991) 216.
[2] K. Drühl, R.G. Wenzel, J.L. Carlsten, Phys. Rev. Lett. 51 (1983) 1171.
[3] D.E. Gakhovich, A.S. Grabchikov, V.A. Orlovich, Opt. Commun. 102 (1993) 485..
[4] J.C. Englund, C.M. Bowden, Phys. Rev. Lett. 57 (1986) 2661;
M.G. Raymer, Z.W. Li, I.A. Walmsley Phys. Rev. Lett. 63 (1989) 1586;
D.C. MacPherson, R.C. Swanson, J.L. Carlsten, Phys. Rev. A 40 (1989) 6745.
[5] F.Y.F. Chu, A.C. Scott, Phys. Rev. A 12 (1975) 2060.
[6] H. Steudel, Physica D 6 (1983) 155;
D.J. Kaup, Physica D 6 (1983) 143; Physica D 19 (1986) 125;
K. Drühl, G. Alsing, Physica D 20 (1986) 429;
C.R. Menyuk, Phys. Rev. Lett. 62 (1989) 2937;
D.J. Kaup, C.R. Menyuk, Phys. Rev. A 42 (1990) 1712.
[7] C. Claude, J. Leon, Phys. Rev. Lett. 74 (1995) 3479;
C. Claude, F. Ginovart, J. Leon, Phys. Rev. A 52 (1995) 767.
[8] I.R. Gabitov, V.E. Zakharov, A.V. Mikhailov, Theor. Math. Phys. 63 (1985) 328;
J. Leon, Phys. Rev. A 47 (1993) 3264.
[9] A. Yariv, Quantum Electronics (Wiley, New York, 1975);
A.C. Newell, J.V. Moloney, Nonlinear Optics (Addison-Wesley, Redwood City, CA, 1992).
[10] B.M. Levitan, I.S. Sargsjan, Introduction to Spectral Theory, Transl. Math. Monographs (A.M.S., Providence, 1975);
I.D. Iliev, E.Kh. Khristov, K.P. Kirchev, Spectral Methods in Soliton Equations, Pitman Monographs (Longman, Harlow, UK, 1994).
[11] A.S. Fokas, C.R. Menyuk, J. Nonlinear Sci. (1998), in press.
[12] C.R. Menyuk, D. Levi, P. Winternitz, Phys. Rev. Lett. 69 (1992) 3048;
D. Levi, C.R. Menyuk, P. Winternitz, Phys. Rev. A 44 (1991) 6057.
[13] D.J. Kaup, Physica D 6 (1983) 143.
[14] A.S. Fokas, A.R. Its, Phys. Rev. Lett. 68 (1992) 3117; SIAM J. Math. Anal. 27 (1996) 738.
[15] H.Steudel, Quantum Opt. 2 (1990) 387.
[16] R.L. Chou, C.K. Chu, Phys. Fluids A 2 (1190) 1574;
C.K. Chu, R.L. Chou, Adv. Appl. Mech. 27 (1990) 283.
[17] J. Leon, J. Math. Phys. 35 (1994) 3504.