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Resonant two-wave interaction in the Davydov model

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Abstract
The Davydov model for exciton–phonon coupling in hydrogen-bonded molecular chains is reconsidered in the context of two-wave resonant interaction. By applying a semi-discrete slowly varying envelope approximation, when the physical problem is that of the long-distance evolution of an input finite duration excitonic pulse, we derive an integrable limit model which preserves the coupling nature of the process. The spectral transform is constructed with emphasis on the complete characterization of the spectral data. As an application, the localized one-soliton solution is explicitly constructed. Then by using Darboux–Bäcklund transformations, a non-local (or topological) one-soliton solution is also derived. As a consequence, the system possess two different soliton solutions where the phonon component is a localized pulse, but where the exciton wave is either localized (bell shape) or topological (kink shape). The resulting approximate soliton solutions of the Davydov model in the resonant regime are subsonic in the localized case and supersonic in the topological case. Finally, by expressing the Bianchi superposition theorem, a nonlinear superposition formula is derived allowing for explicit two-soliton solution.

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1. Introduction

The Davydov model [1], later modified and largely studied by Scott [2], describes the coupling between excitons and phonons in a diatomic molecular chain. Well-known physical situations where this model applies are the alpha helix proteins and the acetanilide molecule. They have the common property of being constituted by one-dimensional chains of hydrogen-bonded peptide groups, supporting the phonon wave, that can couple to excitations of the amide-I (C═O stretching). In the Scott version, the amide-I couples only to the single adjacent hydrogen bond elongation while in the Davydov approach it is coupled to the two nearest H-bonds.

The Davydov model of such a coupling results in the following system of coupled equations [2, 3],

$$\begin{aligned} i\hbar\dot{a}_n &= [E_0 + W + \chi(\beta_{n+1} - \beta_n)]a_n - J(a_{n+1} + a_{n-1}) \\ M\ddot{\beta}_n &= K(\beta_{n+1} - 2\beta_n + \beta_{n-1}) + \chi(|a_n|^2 - |a_{n-1}|^2) \end{aligned} \tag{1.1}$$

written here for a single chain. The state of the amide-I excitation is described by the eigenfunction $a_n(t)$ solution of a time-dependent Schrödinger equation in the *potential* $\beta_{n+1} - \beta_n$. The dynamical variable β_n stands for the longitudinal displacement along the chain, M is the mass of the peptide group, K is the spring constant of the hydrogen bond, E_0 is the energy of the amide-I and W is the total energy of the peptide group displacements. The constant χ is the exciton–phonon coupling parameter and J measures the energy of the dipole–dipole interaction of amide-I oscillations. An overdot stands for derivation with respect to time T.

Upon defining the new dimensionless time variable $t = JT/\hbar$, and the adimensional quantities

$$\Psi_n = a_n \frac{\hbar\chi}{J\sqrt{JM}} \exp\left[\frac{i}{J}(E_0 + W - 2J)t\right] \qquad Q_n = \frac{\chi}{J}(\beta_{n+1} - \beta_n) \tag{1.2}$$

the system (1.1) becomes

$$i\partial_t\Psi_n + (\Psi_{n+1} - 2\Psi_n + \Psi_{n-1}) = Q_n\Psi_n \tag{1.3}$$

$$\partial_t^2 Q_n - V^2(Q_{n+1} - 2Q_n + Q_{n-1}) = |\Psi_{n+1}|^2 - 2|\Psi_n|^2 + |\Psi_{n-1}|^2. \tag{1.4}$$

The only remaining physical constant is the *adimensional sound velocity*

$$V = \frac{\hbar}{J}v_p \qquad v_p = \sqrt{\frac{K}{M}} \tag{1.5}$$

where v_p denotes the phonon velocity in units of cells per second. Note that the coupling parameter χ enters now in the amplitude of $\Psi_n(t)$.

Our basic system (1.3), (1.4) applies to a number of physically interesting phenomena involving exciton–phonon or electron–phonon coupling [4, 5]. Moreover, interesting numerical studies have been performed recently [6] where a kind of resonance phenomena has been unveiled. We understand it here as a *two-wave resonant interaction* and then study the properties of the system in the region of parameters where this resonance occurs. Note that the continuous limit of the model (1.3), (1.4) is nothing but the Zakharov equation governing the Langmuir–acoustic wave interaction in a plasma [7].

To be more precise, we are interested in the long-distance effect of the scattering of an excitonic wave packet Ψ_n with resonant carrier frequency (see definition later) entering a medium at $n = 0$ where coupling with the phonon wave is allowed for by the system (1.3), (1.4). In particular, we shall derive the following limit system,

$$\frac{i}{2}u_y + u_{ss} = qu \qquad q_y = 4(|u|^2)_s \tag{1.6}$$

where u and q represent the slowly varying envelopes of the excitonic and of the phonon waves, respectively, and where the variable s stands for a scaled retarded time varying on $\mathbb{R}$, and y for a scaled space extending on any finite support $y \in [0, L]$.

The above system was shown to be integrable in [8], where it was associated with a 3×3 first-order spectral problem of the Zakharov–Shabat type and solved with the traditional Gel'fand–Levitan–Marchenko approach for *potentials* q and u vanishing at large s. Soliton solutions vanishing at infinity were also obtained. Here, we reconsider the spectral transform with two main objectives: first, the solution of the characterization problem for the spectral data, and second, the solution of the inverse problem in terms of the Cauchy–Green solution of a Riemann–Hilbert boundary-value problem.

To this end we found it convenient to consider the more general system of nonlinear equations

$$q_y = 4(uv)_s \qquad \frac{\mathrm{i}}{2}u_y + u_{ss} = qu \qquad -\frac{\mathrm{i}}{2}v_y + v_{ss} = qv \tag{1.7}$$

which reduces to (1.6) for

$$v = \overline{u} \qquad q \in \mathbb{R} \tag{1.8}$$

and to relate it to the following mixed Schrödinger/Zakharov–Shabat system,

$$\varphi_{1,ss} + k^2\varphi_1 - q\varphi_1 = -\mathrm{i}u\varphi_2 \qquad \varphi_{2,s} = v\varphi_1 \tag{1.9}$$

which, in the reduced case, is equivalent to the spectral problem used in [8].

We solve then the direct and inverse spectral problem and derive the soliton solution in the space of functions q, u and v *vanishing* at large s. In the reduced case (1.8), this solution will be shown to be (note that u is defined up to an arbitrary constant phase)

$$u(y,s) = \frac{\eta\sqrt{2\zeta}\,\mathrm{e}^{\mathrm{i}\zeta s - 2\mathrm{i}(\zeta^2-\eta^2)y}}{\cosh(\eta s - 4\zeta\eta y)} \tag{1.10}$$

$$q(y,s) = \frac{-2\eta^2}{\cosh^2(\eta s - 4\zeta\eta y)} \tag{1.11}$$

where η and ζ are the soliton parameters (real valued and with $\zeta > 0$). We recover with this expression the soliton solution found in [8].

However, this is not the only admissible soliton solution of the system and, by constructing the Darboux–Bäcklund transformation, we obtain a one-soliton solution with q vanishing at large s, but with u and v *non-vanishing* at large s. In the reduced case (1.8) it reads

$$u(y,s) = \sqrt{2\zeta}[\mathrm{i}\zeta - \eta\tanh(\eta s + 4\zeta\eta y)] \tag{1.12}$$

$$q(y,s) = \frac{-2\eta^2}{\cosh^2(\eta s + 4\zeta\eta y)} \tag{1.13}$$

with again real-valued parameters η and $\zeta > 0$. Note that these two solitons differ not only in their topological properties but also in their direction of propagation as indeed the parameter ζ is positive in both cases.

Returning to the physical quantities, these solitons provide the following approximate solutions of the original system. In the case of a localized soliton, the approximate solution reads

$$\Psi_n(t) = \mathrm{e}^{\mathrm{i}(k_r n - \omega_r t)}V\sqrt{2\zeta}\left(\frac{2\sqrt{\cos k_r}}{\sin k_r}\right)^{1/2}\frac{\eta}{\cosh Z_n(t)} \tag{1.14}$$

$$Q_n(t) = \frac{-2\eta^2}{\cosh^2 Z_n(t)} \tag{1.15}$$

$$Z_n(t) = \frac{\eta V}{\sqrt{\cos k_r}}\left(t - \frac{n}{V}\right) - \frac{\eta\zeta}{\sin k_r}(n - n_0) \tag{1.16}$$

where the initial position n_0 is arbitrary and the resonant wave number k_r is defined in (2.6) by $V = 2\sin k_r$ while the resonant frequency results from the dispersion relation $\omega_r = 2(1-\cos k_r)$. Note that the velocity of the pulse is given by

$$v = V\left(1+\zeta\frac{\sqrt{\cos k_r}}{\sin k_r}\right)^{-1} \tag{1.17}$$

which means a *subsonic* solitary wave ($v < V$). Indeed, as discussed later the parameter ζ is a small (positive) quantity, and the value of k_r is close to, and less than $\pi/2$.

Next, in the case of non-localized solution $u(y,s)$ the approximate solution reads

$$\Psi_n(t) = \mathrm{e}^{\mathrm{i}(k_r n-\omega_r t)}V\sqrt{2\zeta}\left(\frac{2\sqrt{\cos k_r}}{\sin k_r}\right)^{1/2}[\mathrm{i}\zeta - \eta\tanh X_n(t)] \tag{1.18}$$

$$Q_n(t) = \frac{-2\eta^2}{\cosh^2 X_n(t)} \tag{1.19}$$

$$X_n(t) = \frac{\eta V}{\sqrt{\cos k_r}}\left(t-\frac{n}{V}\right)+\frac{\eta\zeta}{\sin k_r}(n-n_0) \tag{1.20}$$

which is a *supersonic* solitary wave as indeed

$$v = V\left(1-\zeta\frac{\sqrt{\cos k_r}}{\sin k_r}\right)^{-1}. \tag{1.21}$$

These expressions are approximate solutions to the Davydov system in the precise sense given in (2.30). Their role in a given physical context is reported in future work.

2. Resonant interaction, integrable limit

2.1. Two-wave resonant process

Following [9, 10], two-wave resonant interaction can be understood as a three-wave interaction between two excitonic waves (ω_1, k_1), (ω_2, k_2), and one phonon wave (Ω, K) in the limit $K \to 0$ for which $k_1 \to k_2$. From the selection rules of the three-wave process

$$k_1 - k_2 = K \qquad \omega(k_1) - \omega(k_2) = \Omega(K) \tag{2.1}$$

we can write

$$\frac{\omega(k_1)-\omega(k_2)}{k_1-k_2} = \frac{\Omega(K)}{K} \tag{2.2}$$

which in the limit $K \to 0$, that is $k_1 \to k_2$, leads to the two-wave resonant interaction criterion

$$\left.\frac{\partial\omega(k)}{\partial k}\right|_{k_r} = \lim_{K\to 0}\frac{\Omega(K)}{K} \tag{2.3}$$

where we have defined the resonant wave number $k_r = k_1 = k_2$.

A carrier wave $\exp[\mathrm{i}(kn-\omega t)]$ leads to the linear dispersion relation of (1.3)

$$\omega(k) = 2(1-\cos k) \tag{2.4}$$

while the phonon wave dispersion relation reads

$$\Omega^2(K) = 2V^2(1-\cos K). \tag{2.5}$$

The solution $k_r \in [0,\pi]$ of the resonant condition (2.3) determines the value of the resonant frequency $\omega(k_r) = \omega_r$ of the exciton wave and we get from the above dispersion relations

$$\sin k_r = \tfrac{1}{2}V \qquad \omega_r = 2(1-\cos k_r). \tag{2.6}$$

We expect this resonance to be a candidate for the interpretation of the anomalous absorption band at 1650 cm^{-1} discovered in crystalline acetanilide [11]. In such a situation the amide-I lies at $E_0/hc = 1665\ \mathrm{cm}^{-1}$, hence the shift $\mathrm{d}E$ is of order 15 cm^{-1}. Returning to the physical time, the resonant frequency ω_r gives the energy $\mathrm{d}E = \hbar\omega_r$ which in cm^{-1} reads

$$\mathrm{d}E = 2J'(1-\cos k_r) \qquad J' = \frac{J}{hc} \tag{2.7}$$

(c is the velocity of light in cm s^{-1} and hence J' is nothing but J read in cm^{-1}). Recent experiments have demonstrated that, as conjectured before, the anomalous absorption band is indeed resulting from a nonlinear process as the *anomalous peak is strongly anharmonic* [12].

The wave number k_r is given by equation (2.6), namely from (1.5) by

$$\sin k_r = \frac{v_p}{4\pi c J'}. \tag{2.8}$$

Using numerical values (v_p results from the measurements of the velocity of sound in [13]) we obtain, by solving the system (2.7), (2.8) for $\{J', k_r\}$,

$$\left.\begin{array}{l} v_p = 373\times 10^{10}\ \mathrm{s}^{-1} \\ \mathrm{d}E = 15\ \mathrm{cm}^{-1} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} k_r = 1.30 \\ J' = 10.3\ \mathrm{cm}^{-1} \end{array}\right. \tag{2.9}$$

and thus J lies in a reasonable range (see e.g. [2, p 16]).

As a consequence, the above two-wave interaction process will take place for a value of the resonant wave number $k_r \sim 1.3$, which lies around the centre of the Brillouin zone. This fact forbids the use of the continuous limit approximation for the exciton wavefunction Ψ_n, and we shall proceed now with a *semi-discrete* slowly varying envelope approximation in the spirit of [14]. In all what follows we shall assume then

$$k_r < \frac{\pi}{2} \tag{2.10}$$

as the value $\pi/2$ gives rise to singularities (for which a particular treatment would be required).

2.2. Modellization of the scattering process

The process we are interested in is the long-distance effect of the scattering of an excitonic wave packet Ψ_n with carrier frequency ω_r (and wave number k_r), entering a medium at $n = 0$ where coupling with the phonon wave is allowed for. The model under consideration is the Davydov system (1.3), (1.4) and, based on the arguments of [15], the wave packet to consider is

$$\Psi_n(t) = \int \mathrm{d}\omega\, \widehat{\Psi}(\omega)\, \mathrm{e}^{\mathrm{i}(kn-\omega t)} \tag{2.11}$$

where the wave number is expanded in terms of the frequency as

$$\omega = \omega_r + \epsilon\nu \qquad k = k_r + \epsilon\frac{\nu}{V} + \epsilon^2 R\nu^2 + \cdots. \tag{2.12}$$

We have from (2.4) and (2.6)

$$V = \left.\frac{\partial\omega(k)}{\partial k}\right|_{k_r} \qquad 2R = \left.\frac{\partial^2 k}{\partial\omega^2}\right|_{k_r}. \tag{2.13}$$

Note that the group velocity at resonance V is given in (2.6) and is nothing but the adimensional phonon velocity. Note also that the dispersion coefficient R is given, for $\sin k_r > 0$, by the expression

$$2R = -\frac{\cos k_r}{4\sin^3 k_r} \tag{2.14}$$

resulting from (2.4).

The small parameter ϵ measures the departure of ω from the resonant frequency ω_r. The above wave packet can then be written as

$$\Psi_n(t) = \mathrm{e}^{\mathrm{i}(k_r n - \omega_r t)} f(\xi, \tau) \tag{2.15}$$

with the small-amplitude slowly varying modulation f in the frame

$$\tau = \epsilon \left(t - \frac{n}{V} \right) \qquad \xi = \epsilon^2 n. \tag{2.16}$$

This frame indicates that we consider long (ϵ^{-2}) distance effects in the retarded slow (ϵ^{-1}) time allowing the input disturbance enough time to reach the observer, which indeed corresponds to the scattering of a wave packet.

2.3. Asymptotic model

Resulting from these general considerations, we seek now a solution of the system (1.3), (1.4) under the following form:

$$\Psi_n(t) = \epsilon^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(k_r n - \omega_r t)} \sum_{j=1}^{\infty} \epsilon^j f^{(j)}(\xi, \tau) \tag{2.17}$$

$$Q_n(t) = \epsilon \sum_{j=1}^{\infty} \epsilon^j q^{(j)}(\xi, \tau). \tag{2.18}$$

Note that the envelope of the exciton wave is continuous but the carrier is kept discrete. Note also the relative scaling between Ψ_n and Q_n which results from a balance in the coupling terms of the model.

We consider the above expansions for Ψ_n and Q_n up to the order $\epsilon^{7/2}$ and ϵ^5, respectively, i.e.

$$\begin{aligned} \Psi_{n+m} = \epsilon^{3/2} \mathrm{e}^{\mathrm{i}k_r(n+m) - \mathrm{i}\omega_r t} \Bigg[& \left(1 - \frac{\epsilon m}{V} \partial_\tau + \epsilon^2 m \partial_\xi + \epsilon^2 \frac{m^2}{2V^2} \partial_\tau^2 \right) f^{(1)} \\ & + \epsilon \left(1 - \epsilon \frac{m}{V} \partial_\tau \right) f^{(2)} + \epsilon^2 f^{(3)} \Bigg] + \mathcal{O}(\epsilon^{9/2}) \end{aligned} \tag{2.19}$$

$$\begin{aligned} Q_{n+m} = \epsilon^2 & \left(1 - \epsilon \frac{m}{V} \partial_\tau + \epsilon^2 m \partial_\xi + \epsilon^2 \frac{m^2}{2V^2} \partial_\tau^2 - \epsilon^3 \frac{m^2}{V} \partial_\tau \partial_\xi - \epsilon^3 \frac{m^3}{6V^3} \partial_\tau^3 \right) q^{(1)} \\ & + \epsilon^3 \left(1 - \epsilon \frac{m}{V} \partial_\tau + \epsilon^2 m \partial_\xi + \epsilon^2 \frac{m^2}{2V^2} \partial_\tau^2 \right) q^{(2)} \\ & + \epsilon^4 \left(1 - \epsilon \frac{m}{V} \partial_\tau \right) q^{(3)} + \epsilon^5 q^{(4)} + \mathcal{O}(\epsilon^6). \end{aligned} \tag{2.20}$$

By inserting these expressions into (1.3) we obtain at the order $\epsilon^{3/2}$ the dispersion relation (2.4), at the order $\epsilon^{5/2}$ the resonant condition (2.3) and finally at the order $\epsilon^{7/2}$

$$2\mathrm{i} \sin k_r f_\xi + \frac{\cos k_r}{V^2} f_{\tau\tau} = qf \tag{2.21}$$

where we have defined

$$f = f^{(1)}(\xi, \tau) \qquad q = q^{(1)}(\xi, \tau). \tag{2.22}$$

Next, the phonon equation at the order ϵ^5 leads to the equation

$$q_{\xi\tau} = \frac{1}{2V^3}(|f|^2)_{\tau\tau}. \tag{2.23}$$

It is worth remarking that such a limit works because of the resonant condition (2.3), $\sin k_r = V/2$. This can then be thought of as a *result* of the requirement of a constructive interaction of fields of structure (2.17), (2.18). Indeed, examination of the lowest orders in ϵ in the equation for the phonons, for slow variables as in (2.16) but with the group velocity v_g instead of V, would lead precisely to the resonant criterion $v_g = V$.

In summary, upon integration of (2.23), we have obtained the following system:

$$2\mathrm{i}\sin k_r f_\xi + \frac{\cos k_r}{V^2} f_{\tau\tau} = qf \qquad q_\xi = \frac{1}{2V^3}(|f|^2)_\tau. \tag{2.24}$$

As expected, this system is invariant under the reverse transformation of field and variables that gets rid of the small parameter ϵ, namely

$$\tau \to \tau\epsilon^{-1} \qquad \xi \to \xi\epsilon^{-2} \qquad f \to f\epsilon^{-3/2} \qquad q \to q\epsilon^{-2}. \tag{2.25}$$

The above step can actually be used as a means to check the relative scaling. Then the system (2.24) will be considered from now on as written in the physical (still adimensional) world.

Finally, the constants that still appear in (2.24) can be scaled off by the change of variables, remember (2.10),

$$y = \frac{\xi}{4\sin k_r} \qquad s = \frac{V\tau}{\sqrt{\cos k_r}} \qquad u(y,s) = \frac{1}{V}\left(\frac{\sin k_r}{2\sqrt{\cos k_r}}\right)^{1/2} f(\xi,\tau) \tag{2.26}$$

for which it reads as announced in the introduction, namely

$$\frac{\mathrm{i}}{2}u_y + u_{ss} = qu \qquad q_y = 4(|u|^2)_s. \tag{2.27}$$

It is worth remembering that $u(y,s)$ represents the envelope of the excitonic wave while $q(y,s)$ stands for the envelope of the phonon wave where the time s takes values on the real line and where the space y lies on any given finite interval.

2.4. Approximate solutions

From the preceding various changes of variables, we may now transform the one-soliton solution, for instance (1.12), (1.13), to the corresponding fields in the physical space. The approximate solution (1.18), (1.19) represents then the first terms in expansions (2.17) and (2.18). Hence the value of $Q_n(t)$ has a precision $\mathcal{O}(\epsilon^3)$ while $\Psi_n(t)$ holds to $\mathcal{O}(\epsilon^{5/2})$.

The value of ϵ is specified by the choice of the boundary value $\Psi_0(t)$ as a measure of the relative variations of phase of the envelope versus the carrier. For instance, in expression (1.18) for Ψ_n, we receive

$$\epsilon \sim \eta \tag{2.28}$$

which does fit the scaling of time in (2.16) and moreover, is consistent with the amplitude ϵ^2 of the dominant term in Q_n. Looking then at the spatial dependence in (2.16) we obtain

$$\eta\zeta \sim \epsilon^2 \quad \Rightarrow \quad \zeta \sim \epsilon \tag{2.29}$$

which again is consistent with the amplitude $\epsilon^{3/2}$ of the dominant term in Ψ_n.

To summarize, we thus may write

$$\begin{aligned} \Psi_n(t) &= \mathrm{e}^{\mathrm{i}(k_r n - \omega_r t)} V\sqrt{2\zeta}\left(\frac{2\sqrt{\cos k_r}}{\sin k_r}\right)^{1/2} [\mathrm{i}\zeta - \eta\tanh X_n(t) + \mathcal{O}(\eta^2)] \\ Q_n(t) &= \frac{-2\eta^2}{\cosh^2 X_n(t)} + \mathcal{O}(\eta^3) \end{aligned} \tag{2.30}$$

where the argument $X_n(t)$ is defined in (1.20). The same relations hold also for the local approximate solution (1.14), (1.15).

3. The inverse spectral transform

3.1. Boundary-value problem

We are interested here in the evolution along N molecules of a chain, i.e. $y \in [0, L]$, of a signal sent in $n = 0$. In other words we assume the following data,

$$t \in \mathbb{R} : \{\Psi_0(t), Q_0(t)\} \tag{3.1}$$

which map through the change of variables to

$$s \in \mathbb{R} : \{u(0, s) = u_0(s), q(0, s) = q_0(s)\}. \tag{3.2}$$

The method of the inverse spectral (scattering) transform (IST) will be used to solve the system (1.6) on the domain $y \in [0, L]$ and $s \in (-\infty, +\infty)$ associated with the Dirichlet boundary-value data of $u_0(s)$ and $q_0(s)$. Note that with respect to the usual language in the inverse scattering theory, s plays the role of the space variable and y the role of time.

The IST is considered in the space of functions vanishing sufficiently rapidly as $s \to \pm\infty$. We shall see that the evolution in y does not destroy the good properties of the spectral data ensuring that $u(y, s)$ and $q(y, s)$ preserve their good behaviour at large s for any y, as usual in the inverse scattering theory [16].

3.2. Lax pair

We consider a Lax pair of linear operators defining the mixed Schrödinger and Zakharov–Shabat spectral problem

$$\varphi_{1,ss} + k^2\varphi_1 - q\varphi_1 = -\mathrm{i}u\varphi_2 \qquad \varphi_{2,s} = v\varphi_1 \tag{3.3}$$

and the auxiliary spectral problem

$$\varphi_{1,y} = -\mathrm{i}k^2\varphi_1 + 2u\varphi_2 \qquad \varphi_{2,y} = \mathrm{i}k^2\varphi_2 + 2\mathrm{i}v\varphi_{1,s} - 2\mathrm{i}v_s\varphi_1 \tag{3.4}$$

where k is the *spectral parameter* and q, u and v are the *potentials*. The compatibility of these two systems leads to the following y-evolution equations:

$$q_y = 4(uv)_s \qquad \frac{\mathrm{i}}{2}u_y + u_{ss} = qu \qquad -\frac{\mathrm{i}}{2}v_y + v_{ss} = qv. \tag{3.5}$$

This integrable nonlinear system reduces to our equation (1.6) for

$$v = \overline{u} \qquad q \in \mathbb{R}. \tag{3.6}$$

Solving the *direct problem* for the above Lax pair means using the spectral problem (3.3) to define the *spectral data* as a set of three functions of the spectral parameter k. Then, the auxiliary spectral problem (3.4) is used to obtain the dependence of these spectral data on the variable y. Solving then the *inverse problem* results in the reconstruction of the potentials $q(y, s)$, $u(y, s)$ and $v(y, s)$, at fixed arbitrary value of y, from the spectral data.

In the following, when convenient, we shall use the vectorial notation

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \tag{3.7}$$

for any solution of the spectral problems (3.3) and (3.4).

3.3. Jost solutions

We define the following two sets of Jost solutions, namely

$$\begin{aligned}\varphi_1^\pm &= \mathrm{e}^{-\mathrm{i}ks} + \int_{\mp\infty}^{s} \mathrm{d}s' \frac{\mathrm{e}^{\mathrm{i}k(s-s')} - \mathrm{e}^{-\mathrm{i}k(s-s')}}{2\mathrm{i}k} \left[q\varphi_1^\pm - \mathrm{i}u\varphi_2^\pm\right] \\ \varphi_2^\pm &= \int_{\mp\infty}^{s} \mathrm{d}s' v\varphi_1^\pm\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\psi_1^\pm &= \int_{\mp\infty}^{s} \mathrm{d}s' \frac{\mathrm{e}^{\mathrm{i}k(s-s')}}{2\mathrm{i}(k \pm \mathrm{i}0)} \left[q\psi_1^\pm - \mathrm{i}u\psi_2^\pm\right] - \int_{\pm\infty}^{s} \mathrm{d}s' \frac{\mathrm{e}^{-\mathrm{i}k(s-s')}}{2\mathrm{i}(k \pm \mathrm{i}0)} \left[q\psi_1^\pm - \mathrm{i}u\psi_2^\pm\right] \\ \psi_2^\pm &= 1 + \int_{+\infty}^{s} \mathrm{d}s' v\psi_1^\pm\end{aligned} \tag{3.9}$$

where the signs $\pm$ are in correspondence. Note that for $u = 0$ the Jost solutions in (3.8) reduce to the traditional Jost solutions of the Schrödinger spectral problem.

These four Jost solutions $\varphi^\pm$, $\psi^\pm$ are not independent since the spectral problem (3.3) is of third order and in fact the symmetry

$$\psi^+(k) = \psi^-(-k) \tag{3.10}$$

follows directly from the definition of ψ.

The integral equations defining the Jost solutions $\varphi^\pm$ can be studied using the same method used in the case of the Schrödinger spectral problem. One can prove that for potentials satisfying

$$\int_{-\infty}^{+\infty} \mathrm{d}s(1 + |s|) \begin{pmatrix} |q(s)| \\ |u(s)| \\ |v(s)| \end{pmatrix} < +\infty \tag{3.11}$$

the Jost solutions $\varphi^+ \mathrm{e}^{\mathrm{i}ks}$ (respectively $\varphi^- \mathrm{e}^{\mathrm{i}ks}$) are bounded and analytic in the upper (respectively lower) half complex plane of the spectral parameter k. The integral equation defining ψ^+ requires unusual analysis, reported in the appendix. As a result for potentials satisfying (3.11) ψ^+ is bounded and analytic in the upper half complex plane of k, once a factorizing singularity $1/(k + \mathrm{i}0)$ is subtracted and possible poles.

3.4. Direct problem

As usual in the inverse scattering theory, we consider $\varphi^\pm$ and $\psi^\pm$ as unique analytical functions defined as φ^+ and ψ^+ in the upper half plane and as φ^- and ψ^- in the lower half plane and we evaluate their discontinuity along the real k-axis. This is done by direct use of the defining integral equations. In doing this it is convenient to use instead of $\varphi^+(k, s)$ the eigenfunction $\phi^+(k, s)$ defined by the following integral equation,

$$\begin{aligned}\phi_1^+(k, s) &= \mathrm{e}^{-\mathrm{i}ks} + \frac{1}{2\mathrm{i}(k + \mathrm{i}0)} \int_{-\infty}^{s} \mathrm{d}s' \, \mathrm{e}^{\mathrm{i}k(s-s')} \left[-\mathrm{i}u\phi_2^+ + q\phi_1^+\right] \\ &\quad + \frac{1}{2\mathrm{i}(k + \mathrm{i}0)} \int_{s}^{+\infty} \mathrm{d}s' \, \mathrm{e}^{-\mathrm{i}k(s-s')} \left[-\mathrm{i}u\phi_2^+ + q\phi_1^+\right] \\ \phi_2^+(k, s) &= \int_{-\infty}^{s} \mathrm{d}s' v\phi_1^+\end{aligned} \tag{3.12}$$

which has a slightly different kernel with respect to $\varphi^+(k,s)$ and is simply related to it. In fact, by simple comparison of (3.8) with (3.12) we obtain that

$$\phi^+(k,s) = \frac{1}{a^+(k)}\varphi^+(k,s) \tag{3.13}$$

where the coefficient a^+, usually called a Jost function, is

$$a^+(k) = 1 - \frac{1}{2\mathrm{i}(k+\mathrm{i}0)}\int_{-\infty}^{+\infty} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}ks}\left[q(s)\varphi_1^+(k,s) - \mathrm{i}u(s)\varphi_2^+(k,s)\right]. \tag{3.14}$$

The analytic properties of $\varphi^+(k,s)$ imply that $a^+(k)$ is analytic in the upper half plane, where it may possess isolated zeros, which for simplicity we suppose to be finite in number and simple. Therefore ϕ^+ may have simple poles at $k = k_j$ $(j = 1, 2, \ldots, N)$. Correspondingly, we suppose that also the poles of ψ^+ are simple and N in number, and, precisely, located at $k = \widehat{k}_j$ $(j = 1, 2, \ldots, N)$. These poles are related, as we will see in the following, to the presence of solitonic structures in the solution.

For notation coherence we rename $\varphi^-(k,s)$ as $\phi^-(k,s)$. By use of the integral equations defining the Jost solutions ϕ and ψ we obtain for their discontinuity across the real k-axis the following coupled Riemann–Hilbert relations,

$$\phi^+(k,s) - \phi^-(k,s) = \alpha^+(k)\psi^+(k,s) + \beta^+(k)\phi^-(-k,s) \tag{3.15}$$

$$\psi^+(k,s) - \psi^-(k,s) = \gamma^+(k)\phi^-(-k,s) - \gamma^+(-k)\phi^-(k,s) \tag{3.16}$$

where

$$\alpha^\pm(k) = \int_{-\infty}^{+\infty} \mathrm{d}s\, v(s)\phi_1^\pm(k,s) \tag{3.17}$$

$$\gamma^\pm(k) = \int_{-\infty}^{+\infty} \mathrm{d}s \frac{\mathrm{e}^{-\mathrm{i}ks}}{2\mathrm{i}(k\pm\mathrm{i}0)}\left[q(s)\psi_1^\pm(k,s) - \mathrm{i}u(s)\psi_2^\pm(k,s)\right] \tag{3.18}$$

and

$$\beta^\pm(k) = \rho^\pm(k) - \alpha^\pm(k)\gamma^\pm(k) \tag{3.19}$$

with

$$\rho^\pm(k) = \int_{-\infty}^{+\infty} \mathrm{d}s \frac{\mathrm{e}^{-\mathrm{i}ks}}{2\mathrm{i}(k\pm\mathrm{i}0)}\left[q(s)\phi_1^\pm(k,s) - \mathrm{i}u(s)\phi_2^\pm(k,s)\right]. \tag{3.20}$$

The coefficients α^+, β^+ and γ^+ are the spectral data related to the continuous spectrum ($k \in \mathbb{R}$). Their explicit expressions given above in terms of the Jost solutions and potentials solve the direct spectral problem for the continuous spectrum. Note that, due to the factor $1/(k+\mathrm{i}0)$, β^+ and γ^+ are tempered distributions. For future convenience also the alternative, and of course equivalent, set of spectral data α^-, β^- and γ^- were introduced. They can be obtained analogously modifying the integral equation defining φ^- and leaving unchanged that defining φ^+.

Before turning to the discrete spectrum, it is worth remarking that these two sets of spectral data determine the asymptotic behaviour of the Jost solutions at $s = \pm\infty$. Precisely, by direct use of the integral equations for $\phi^{\pm}$ and ψ^+ and recalling the definitions (3.17), (3.18), (3.20), (3.13) and (3.14) we obtain at $s = +\infty$

$$\begin{aligned}
\phi^+(k,s) &\underset{s\to+\infty}{\sim} \begin{pmatrix} e^{-iks} + \rho^+ e^{iks} \\ \alpha^+ \end{pmatrix} && \begin{cases} k \in \mathbb{R} \\ k \in \mathbb{R} \end{cases} \\
\phi^-(k,s) &\underset{s\to+\infty}{\sim} \begin{pmatrix} e^{-iks} \\ 0 \end{pmatrix} && \begin{cases} k_{\mathrm{Im}} < 0 \\ k_{\mathrm{Im}} < 0 \end{cases} \\
\psi^+(k,s) &\underset{s\to+\infty}{\sim} \begin{pmatrix} \gamma^+ e^{iks} \\ 1 \end{pmatrix} && \begin{cases} k \in \mathbb{R} \\ k_{\mathrm{Im}} > 0 \end{cases}
\end{aligned} \tag{3.21}$$

and at $s = -\infty$

$$\begin{aligned}
\phi^+(k,s) &\underset{s\to-\infty}{\sim} \begin{pmatrix} \frac{1}{a^+(k)} e^{-iks} \\ 0 \end{pmatrix} && \begin{cases} k_{\mathrm{Im}} > 0 \\ k_{\mathrm{Im}} > 0 \end{cases} \\
\phi^-(k,s) &\underset{s\to-\infty}{\sim} \begin{pmatrix} a^-(k) e^{-iks} - \rho^-(k) e^{iks} \\ -\alpha^-(k) \end{pmatrix} && \begin{cases} k \in \mathbb{R} \\ k \in \mathbb{R} \end{cases} \\
\psi^+(k,s) &\underset{s\to-\infty}{\sim} \begin{pmatrix} -\gamma^-(-k) e^{-iks} \\ \tau^+(k) \end{pmatrix} && \begin{cases} k \in \mathbb{R} \\ k_{\mathrm{Im}} > 0. \end{cases}
\end{aligned} \tag{3.22}$$

where we introduced the Jost functions

$$\tau^+(k) = 1 - \int_{-\infty}^{+\infty} ds'\, v(s')\psi_1^+(k,s') \tag{3.23}$$

$$a^-(k) = 1 + \frac{1}{2i(k - i0)} \int_{-\infty}^{+\infty} ds\, e^{iks}[q(s)\varphi_1^-(k,s) - iu(s)\varphi_2^-(k,s)] \tag{3.24}$$

which are respectively analytic in the upper and lower half k plane with the exception of possible poles for $\tau^+(k)$. The region in the complex k plane where the asymptotic value is valid is explicitly indicated on the right.

To study then the discrete part of the spectrum we introduce the Wronskian of any three solutions, say φ, ψ and ξ, of the spectral problem (3.3),

$$W(\varphi, \psi, \xi) = \det \begin{pmatrix} \varphi_1 & \psi_1 & \xi_1 \\ \varphi_{1,s} & \psi_{1,s} & \xi_{1,s} \\ \varphi_2 & \psi_2 & \xi_2 \end{pmatrix}. \tag{3.25}$$

One can check that this Wronskian is independent of s. Then, we consider the Wronskian $W(\phi^+(k), \phi^-(-k), \psi^+(k))$ which can be evaluated at any s, in particular for $s \to \infty$, which from (3.21) gives

$$W(\phi^+(k), \phi^-(-k), \psi^+(k)) = 2ik. \tag{3.26}$$

This Wronskian can also be evaluated by using the asymptotic behaviour of the Jost solutions at $s = -\infty$. One gets

$$W(\phi^+(k), \phi^-(-k), \psi^+(k)) = 2ik\tau^+(k)\frac{a^-(-k)}{a^+(k)}. \tag{3.27}$$

Comparing with (3.26) we have

$$\tau^+(k) = \frac{a^+(k)}{a^-(-k)} \tag{3.28}$$

showing that the poles of ψ^+ are related to the zeros of the Jost function $a^-(-k)$. We suppose that the zeros of $a^-(-k)$ are simple and, then, $k_i \neq \widehat{k}_j$ for any i and j.

From (3.26) at the pole $k = k_j$ of ϕ^+ we have

$$\operatorname*{Res}_{k_j}\{\phi^+(k, s)\} = A_j\psi^+(k_j, s) + B_j\phi^-(-k_j, s). \tag{3.29}$$

Now we consider the second component of this vectorial equality, insert in it the integral equations defining the Jost solutions and compute the limit at $s = +\infty$ of both sides recalling (3.21). Since the limit on the rhs is well defined for $k_{\mathrm{Im}} > 0$, we obtain, on the one hand, that the integral

$$c_j^+ = \int_{-\infty}^{+\infty} \mathrm{d}s' v(s') \operatorname*{Res}_{k_j}\left\{\phi_1^+(k, s')\right\} \tag{3.30}$$

is finite and, on the other hand, the following explicit expression of A_j in terms of the Jost solutions and of the potentials

$$A_j = c_j^+. \tag{3.31}$$

Analogously, using (3.22) we deduce taking the limit for $s \to -\infty$ that the integral

$$c_j^- = \int_{-\infty}^{+\infty} \mathrm{d}s' v(s')\phi_1^-(-k_j, s') \tag{3.32}$$

is finite and

$$B_j c_j^- = A_j\tau^+(k_j). \tag{3.33}$$

Therefore, since from (3.28) $\tau^+(k_j) = 0$, we have $B_j = 0$.

From (3.26) at the pole $k = \widehat{k}_j$ of ψ^+ we have

$$\operatorname*{Res}_{k=\widehat{k}_j}\{\psi^+(k)\} = \widehat{A}_j\phi^+(\widehat{k}_j) + \widehat{B}_j\phi^-(-\widehat{k}_j). \tag{3.34}$$

Considering the second component of this equality at $s = +\infty$, thanks to (3.21), we get,

$$\widehat{A}_j \int_{-\infty}^{+\infty} \mathrm{d}s\, v(s)\phi_1^+(\widehat{k}_j, s) = 0 \tag{3.35}$$

and, excluding the case in which the integral equals zero and $\widehat{A}_j$ would be left undetermined, $\widehat{A}_j = 0$. Then, from the asymptotic of the first components of (3.34) by (3.21) we deduce that the integral

$$\widehat{g}_j^+ = \int_{-\infty}^{+\infty} \mathrm{d}s \frac{\mathrm{e}^{-\mathrm{i}\widehat{k}_j s}}{2\mathrm{i}\widehat{k}_j}\left[q(s) \operatorname*{Res}_{k=\widehat{k}_j}\psi_1^+(k, s) - \mathrm{i}u(s) \operatorname*{Res}_{k=\widehat{k}_j}\psi_2^+(k, s)\right] \tag{3.36}$$

is finite and

$$\widehat{B}_j = \widehat{g}_j^+. \tag{3.37}$$

Note that since $\psi^+(k) = \psi^-(-k)$ we have that $\psi^-(k)$ has poles at $k = -\widehat{k}_j$ and

$$\operatorname*{Res}_{k=-\widehat{k}_j}\{\psi^-(k)\} = -\operatorname*{Res}_{k=\widehat{k}_j}\{\psi^+(k)\}. \tag{3.38}$$

We conclude that the two formulae

$$\operatorname*{Res}_{k_j}\{\phi^+(k, s)\} = A_j\psi^+(k_j, s) \tag{3.39}$$

$$\operatorname*{Res}_{k=\widehat{k}_j}\{\psi^+(k)\} = \widehat{B}_j\phi^-(-\widehat{k}_j) \tag{3.40}$$

relating the residues at the poles of ϕ^+ and ψ^+ to special values of ψ^+ and ϕ^- together with equations (3.31), (3.37) for the *normalization coefficients* A_j and $\widehat{B}_j$ and (3.17)–(3.20) completely solve the direct problem furnishing continuous and discrete spectral data in terms of Jost solutions and potentials.

3.5. Inverse problem

Solving the inverse problem means to obtain the potentials u, v and q from the spectral data. This is performed in two steps: first, by reconstructing the Jost solutions $\phi^{\pm}$ and $\psi^{\pm}$ and, second, by using their relations with the potentials.

The large k asymptotic of the Jost solutions (obtained by successive integration by part of their defining integral equations), the discontinuity relations (3.15) and (3.16) on the real k-axis and the relations (3.39), (3.40) at the poles of ϕ^{+} and $\psi^{\pm}$, constitute a complete Riemann–Hilbert boundary-value problem. This problem is solved by the following coupled integral equations written in their respective half planes (remember $\operatorname{Im} k_j > 0$ and $\operatorname{Im} \widehat{k}_j > 0$),

$$\mu^{-}(k,s) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{N} \frac{A_j \psi^{+}(k_j)\,\mathrm{e}^{\mathrm{i}k_j s}}{k-k_j} + \frac{1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\lambda}{\lambda - k + \mathrm{i}0} [\alpha^{+}(\lambda)\psi^{+}(\lambda,s)\,\mathrm{e}^{\mathrm{i}\lambda s} + \beta^{+}(\lambda)\mu^{-}(-\lambda,s)\,\mathrm{e}^{2\mathrm{i}\lambda s}] \tag{3.41}$$

$$\psi^{+}(k,s) = \begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{N} \frac{2\widehat{k}_j \widehat{B}_j \mu^{-}(-\widehat{k}_j)\,\mathrm{e}^{\mathrm{i}\widehat{k}_j s}}{k^2-\widehat{k}_j^2} + \frac{1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\lambda}{\lambda - k - \mathrm{i}0} [\gamma^{+}(\lambda)\mu^{-}(-\lambda,s)\,\mathrm{e}^{\mathrm{i}\lambda s} - \gamma^{+}(-\lambda)\mu^{-}(\lambda,s)\,\mathrm{e}^{-\mathrm{i}\lambda s}] \tag{3.42}$$

where

$$\mu^{\pm}(k,s) = \phi^{\pm}(k,s)\,\mathrm{e}^{\mathrm{i}ks}. \tag{3.43}$$

Note that in these equations, since $\psi^{+}(\lambda)$, $\beta^{+}(\lambda)$ and $\gamma^{+}(\lambda)$ contain a factor $\frac{1}{\lambda+\mathrm{i}0}$, the product of two distributions appears. In the first equation the product $\frac{1}{\lambda-k+\mathrm{i}0}\frac{1}{\lambda+\mathrm{i}0}$ is a well-defined distribution regular at $k=0$. In the second equation the product must be defined as

$$\mathcal{D}(\lambda,k) = \lim_{\varepsilon_2\to+0}\lim_{\varepsilon_1\to+0} \frac{1}{\lambda-k-\mathrm{i}\varepsilon_2}\frac{1}{\lambda\pm\mathrm{i}\varepsilon_1} \tag{3.44}$$

where the limits are to be performed in the order indicated. We have

$$\mathcal{D}(\lambda,k) = \lim_{\varepsilon_2\to+0} \left(\frac{1}{\lambda-k-\mathrm{i}\varepsilon_2} - \frac{1}{\lambda\pm\mathrm{i}0}\right)\frac{1}{k+\mathrm{i}\varepsilon_2} \tag{3.45}$$

which shows that $\mathcal{D}(\lambda,k)$ is a well-defined tempered distribution in the two variables λ and k and that the reconstructed ψ^{+} has a factorizing singularity $1/(k+\mathrm{i}0)$, as we stated in the appendix, solving the direct problem.

The solution of the above closed system of integral equations (3.41), (3.42) furnishes the potentials by extracting the leading k-orders, namely

$$q(s) = -2\mathrm{i}\partial_s \mu_1^{-(1)}(s) \qquad v(s) = -\mathrm{i}\mu_2^{-(1)}(s) \qquad u(s) = \mathrm{i}\psi_1^{+(2)}(s) \tag{3.46}$$

where the upper index (ℓ) stands for the coefficient of $k^{-\ell}$ in the expansions at large k of the Jost solutions.

3.6. y-dependence of the spectral data

In the previous section the spectral problem (3.3) has been used to build a bijection between the set of potentials and the set of spectral data, hence furnishing the complete solution of direct and inverse problems. We now need to find the y-dependence of the spectral data for potentials evolving according to (3.5). Since the spectral data are defined via the Jost solutions we first derive the evolution equations in y of the Jost solutions or, equivalently, the y-dependence of the coefficients of a linear combination of the Jost solutions satisfying both spectral problems (3.3) and (3.4).

Because the Jost solutions have functionally independent behaviour at large s it is enough to consider the forms

$$\omega_1(k, y)\phi^{\pm}(k, y, s) \qquad \omega_2(k, y)\psi^{\pm}(k, y, s) \tag{3.47}$$

and determine the evolution in y of ω_1 and ω_2. By inserting them into (3.4) and by considering the limit at large s, we easily obtain

$$\omega_{1,y} = -\mathrm{i}k^2\omega_1 \qquad \omega_{2,y} = \mathrm{i}k^2\omega_2 \tag{3.48}$$

together with the following y-evolution of the spectral data and Jost functions:

$$\alpha^+_y = 2\mathrm{i}k^2\alpha^+ \qquad \rho^+_y = 0 \qquad \gamma^+_y = -2\mathrm{i}k^2\gamma^+ \qquad a^{\pm}_y = 0. \tag{3.49}$$

Hence, finally, the spectral data and Jost functions satisfy the following y-dependence:

$$\alpha^+(k, y) = \mathrm{e}^{2\mathrm{i}k^2 y}\alpha^+(k, 0) \qquad \gamma^+(k, y) = \mathrm{e}^{-2\mathrm{i}k^2 y}\gamma^+(k, 0) \tag{3.50}$$

$$\beta^+(k, y) = \beta^+(k, 0) \qquad a^{\pm}(k, y) = a^{\pm}(k, 0). \tag{3.51}$$

Returning to the Jost solutions, the above results show that $\phi^{\pm}$ are actually solutions of

$$\phi^{\pm}_{1,y} = 2u\phi^{\pm}_2 \qquad \phi^{\pm}_{2,y} = 2\mathrm{i}k^2\phi^{\pm}_2 + 2\mathrm{i}v\phi^{\pm}_{1,s} - 2\mathrm{i}v_s\phi^{\pm}_1 \tag{3.52}$$

while the eigenfunctions $\psi^{\pm}$ solve

$$\psi^{\pm}_{1,y} = -2\mathrm{i}k^2\psi^{\pm}_1 + 2u\psi^{\pm}_2 \qquad \psi^{\pm}_{2,y} = 2\mathrm{i}v\psi^{\pm}_{1,s} - 2\mathrm{i}v_s\psi^{\pm}_1. \tag{3.53}$$

The y-dependences of the normalization coefficients A_j and $\widehat{B}_j$ are simply obtained by substituting the relations (3.39), (3.40) into (3.52) and (3.53) and then by using them again for $\phi^-(-k)$ and $\psi^+(k)$. We have

$$A_j(y) = A_j(0)\,\mathrm{e}^{2\mathrm{i}k_j^2 y} \qquad \widehat{B}_j(y) = B_j(0)\,\mathrm{e}^{-2\mathrm{i}\widehat{k}_j^2 y}. \tag{3.54}$$

This completes with (3.50), (3.51) the y-dependence of the spectral data.

3.7. Localized one-soliton solution

If the continuous spectrum vanishes, we obtain from (3.41) and (3.42) an algebraic system. We consider, for simplicity, the case $N = 1$ in the reduced case $q \in \mathbb{R}$, $u = \overline{v}$. For a decaying and regular soliton the spectral data must satisfy the characterization equations

$$\widehat{k}_1 = -\overline{k}_1 \qquad k_{1\mathrm{Re}} < 0 \tag{3.55}$$

$$2\overline{k}_1\widehat{B}_1(0) = -\overline{A_1(0)} \tag{3.56}$$

and, then, the one-soliton solution is given by

$$q = \frac{1}{\Delta^2}\frac{|A_1(0)|^2}{k_{1\mathrm{Re}}}\exp(-2k_{1\mathrm{Im}}(s + 4k_{1\mathrm{Re}}y)) \tag{3.57}$$

$$u = \frac{\mathrm{i}}{\Delta}\overline{A_1(0)}\exp\left(-k_{1\mathrm{Im}}(s + 4k_{1\mathrm{Re}}y) - \mathrm{i}k_{1\mathrm{Re}}s - 2\mathrm{i}\left(k^2_{1\mathrm{Re}} - k^2_{1\mathrm{Im}}\right)y\right) \tag{3.58}$$

with

$$\Delta = 1 - \frac{|A_1(0)|^2}{8k_{1\mathrm{Re}}k^2_{1\mathrm{Im}}}\exp(-2k_{1\mathrm{Im}}(s + 4k_{1\mathrm{Re}}y)). \tag{3.59}$$

The soliton written in (1.10), (1.11) is then obtained by defining $k_{1\mathrm{Re}} = -\zeta$ and $k_{1\mathrm{Im}} = \eta$. Moreover, we have scaled the phase $\mathrm{i}\overline{A_1(0)}/|A_1(0)|$ to the value 1 in $u(y, s)$, thanks to the invariance of the system (1.6). When N solitons are present in the solution the characterization equations (3.55), (3.56) extend to each soliton.

4. Darboux and Bäcklund transformations

We consider now the existence of pure soliton solutions non-vanishing at large s, which is not *a priori* excluded. It is well known that the extension of the inverse scattering theory to non-decaying potentials is a difficult task. Then we rather derive and apply the Darboux transformations for the spectral problem (3.3). This is a well-stated technique [17] for constructing Bäcklund transformations and nonlinear superposition formulae. We shall then derive explicit one-soliton and two-soliton solutions whose components u and v do not vanish at large s.

We consider the principal spectral problem (3.3) with potentials $u(s), q(s), v(s)$ and eigenfunction $\varphi(k, s)$ and forget for a while the y-dependence. The Darboux transformation can be obtained by constructing a gauge transformed $\varphi'(k, s)$ of $\varphi(k, s)$ that satisfies the same spectral problem but with transformed potentials $u'(s), q'(s), v'(s)$. The gauge transformation can be conveniently written as

$$\varphi_1' = \alpha_{11}\varphi_1 + \alpha_{12}\varphi_2 + \beta_1\varphi_{1,s} \tag{4.1}$$

$$\varphi_2' = \alpha_{21}\varphi_1 + \alpha_{22}\varphi_2 + \beta_2\varphi_{1,s} \tag{4.2}$$

where the coefficients α_{ij} and β_j are in general polynomials in k^2 and functions of s to be determined. Here we consider the simplest case when they do not depend on k^2. Then, by using explicitly the principal spectral problem (3.3) both for φ and φ' and by identification of the coefficients of φ_1, φ_2 and $\varphi_{1,s}$ we obtain

$$\varphi_1' = -\frac{\chi_{1,s}}{\chi_1}\varphi_1 + \varphi_{1,s} \qquad \varphi_2' = -C\frac{\chi_2}{\chi_1}\varphi_1 + C\varphi_2. \tag{4.3}$$

where C is an arbitrary constant and the vector χ satisfies the original spectral problem, with $k = h$, h being an arbitrary constant,

$$\chi_{1,ss} + h^2\chi_1 + \mathrm{i}u\chi_2 - q\chi_1 = 0 \qquad \chi_{2,s} = v\chi_1. \tag{4.4}$$

The y-dependence is obtained by requiring that the function χ obeys the auxiliary spectral problem (3.4) written for $k = h$, namely

$$\chi_{1,y} = -\mathrm{i}h^2\chi_1 + 2u\chi_2 \qquad \chi_{2,y} = \mathrm{i}h^2\chi_2 + 2\mathrm{i}v\chi_{1,s} - 2\mathrm{i}v_s\chi_1. \tag{4.5}$$

The transformed potentials are then given in terms of χ by

$$q' = q - 2\left(\frac{\chi_{1,s}}{\chi_1}\right)_s \qquad u' = \frac{1}{C}\left[-u\frac{\chi_{1,s}}{\chi_1} + u_s\right] \qquad v' = -C\frac{\chi_2}{\chi_1}. \tag{4.6}$$

This constitutes the Darboux transformation: given a set of potentials q, u and v and two arbitrary constants C and h, the solution $\chi(s, y)$ of the linear systems (4.4), (4.5) furnishes explicitly the new potentials, solution of (3.5), together with a corresponding eigenfunction.

Note that from (4.6) by eliminating via an integration the vector χ, we obtain the so-called Bäcklund transformation ($I = \int^s \mathrm{d}s'$ and d is a constant of integration)

$$\begin{aligned} &q' + q - \frac{1}{2}[I(q'-q) + d]^2 + \frac{2\mathrm{i}}{C}uv' - 2h^2 = 0 \\ &(uv')_s - (u'v' - uv)C = 0 \\ &2v_s' - v'[I(q'-q) + d] + 2Cv = 0 \end{aligned} \tag{4.7}$$

relating the old potentials q, u and v to the new ones q', u' and v'.

It is worth remarking that for $u = 0$ the relation between q' and q is the Bäcklund transformation for the Korteveg–de Vries equation [16].

4.1. Non-localized one-soliton solution

The game consists in starting from a trivial solution q, u, v of the system (3.5) and, then, in building via the Darboux transformation a new solution q', u', v'. A convenient choice here is

$$q = 0 \qquad u = a \qquad v = b \tag{4.8}$$

where a and b are two complex constants.

First, we solve (4.4) and (4.5) for this choice. If ω is a root of

$$\omega(\omega^2 - h^2) = ab \tag{4.9}$$

a solution is given by

$$\chi_1 = \exp\left[\mathrm{i}\omega\left(s - \omega y - \frac{ab}{\omega^2}y\right)\right] \qquad \chi_2 = \frac{b}{\mathrm{i}\omega}\exp\left[\mathrm{i}\omega\left(s - \omega y - \frac{ab}{\omega^2}y\right)\right] \tag{4.10}$$

and the general solution is a linear combination with constant coefficients of the three solutions obtained considering the three roots of (4.9).

The parameters ab and h^2 and the linear combination have to be chosen in such a way that new solution q', u', v' is regular and satisfies the reduction $u' = \overline{v}'$. This analysis is more easily done if ab and h^2 are parametrized as follows:

$$ab = H^3 + K^3 \qquad h^2 = 3HK. \tag{4.11}$$

Then, the roots of equation (4.9) can be written as

$$\omega_0 = H + K \qquad \omega_\pm = -\tfrac{1}{2}(H + K) \pm \mathrm{i}\tfrac{\sqrt{3}}{2}(H - K). \tag{4.12}$$

We fix ab and h^2 by choosing

$$H, K \in \mathbb{R} \qquad H \neq K \tag{4.13}$$

and consider the special solution

$$\begin{aligned}\chi_1 &= \exp\left[\mathrm{i}\omega\left(s - \omega y - \frac{ab}{\omega^2}y\right)\right] + \exp\left[\mathrm{i}\overline{\omega}\left(s - \overline{\omega} y - \frac{ab}{\overline{\omega}^2}y\right)\right] \\ \chi_2 &= \frac{b}{\mathrm{i}\omega}\exp\left[\mathrm{i}\omega\left(s - \omega y - \frac{ab}{\omega^2}y\right)\right] + \frac{b}{\mathrm{i}\overline{\omega}}\exp\left[\mathrm{i}\overline{\omega}\left(s - \overline{\omega} y - \frac{ab}{\overline{\omega}^2}y\right)\right]\end{aligned} \tag{4.14}$$

where, thanks to (4.13),

$$\omega \equiv \omega_+ = \overline{\omega}_- \qquad \omega_{\mathrm{Im}} \neq 0. \tag{4.15}$$

Then it is straightforward to apply formulae (4.6) to obtain the new solution

$$\begin{aligned} q'(y, s) &= \frac{-2\omega_{\mathrm{Im}}^2}{\cosh^2[\omega_{\mathrm{Im}}(s - 4\omega_{\mathrm{Re}} y)]} \\ u'(y, s) &= -\frac{a}{C}\{\mathrm{i}\omega_{\mathrm{Re}} + \omega_{\mathrm{Im}} \tanh[\omega_{\mathrm{Im}}(s - 4\omega_{\mathrm{Re}} y)]\} \\ v'(y, s) &= 2\omega_{\mathrm{Re}}\frac{C}{a}\{-\mathrm{i}\omega_{\mathrm{Re}} + \omega_{\mathrm{Im}} \tanh[\omega_{\mathrm{Im}}(s - 4\omega_{\mathrm{Re}} y)]\} \end{aligned} \tag{4.16}$$

where we used

$$ab = -2\omega_{\mathrm{Re}}|\omega|^2 \tag{4.17}$$

which is valid for this special choice of H and K.

If we now demand $v' = \overline{u}'$, we get

$$-2\omega_{\mathrm{Re}} = \left|\frac{a}{C}\right|^2. \tag{4.18}$$

Since u' is defined up to a constant phase, without loss of generality we can choose the parameter a/C to be real and, finally, we end up in the following one-soliton solution of (1.6),

$$q'(y,s) = \frac{-2\omega_{\mathrm{Im}}^2}{\cosh^2[\omega_{\mathrm{Im}}(s-4\omega_{\mathrm{Re}}y)]} \tag{4.19}$$

$$u'(y,s) = -\sqrt{-2\omega_{\mathrm{Re}}}\,\{\mathrm{i}\omega_{\mathrm{Re}} + \omega_{\mathrm{Im}}\tanh[\omega_{\mathrm{Im}}(s-4\omega_{\mathrm{Re}}y)]\} \tag{4.20}$$

with ω a free complex parameter with $\omega_{\mathrm{Re}} < 0$. Expressions (1.12) and (1.13) are recovered for $\omega_{\mathrm{Re}} = -\zeta$ and $\omega_{\mathrm{Im}} = \eta$. This soliton propagates to the right and has in q the same profile as the KdV soliton, while u has a kink-like shape.

4.2. Nonlinear superposition

An interesting property of integrable systems is the existence of a nonlinear superposition principle, known also as the Bianchi theorem of permutability, which is quite a useful tool for obtaining new explicit solutions by superimposing known solutions.

The nonlinear superposition formula can be obtained by using the gauge transformation (4.3) rewritten in terms of the old and new transformed solutions. We compose two gauges G_1 and G_2 and impose that they commute, namely that

$$\begin{cases} \varphi \xrightarrow{G_1} \varphi^{(1)} \xrightarrow{G_2} \varphi' \\ \varphi \xrightarrow{G_2} \varphi^{(2)} \xrightarrow{G_1} \varphi'. \end{cases} \tag{4.21}$$

Writing down explicitly that the vector φ' is obtained indifferently following one path or the other and remembering that the elements φ_1, $\partial_s\varphi_1$ and φ_2 are linearly independent, after some algebra we eventually obtain the following nonlinear superposition formula:

$$\begin{aligned} u' &= \frac{u_1u_2}{u} + \frac{u_1u_{2,s} - u_2u_{1,s}}{C_1u_1 - C_2u_2} \\ v' &= \frac{v_1v_2}{v} + 2v_1v_2\frac{C_1v_2 - C_2v_1}{v_{1,s}v_2 - v_{2,s}v_1} \\ q' &= q_1 + q_2 - q + 2\partial_s^2\log\left(\frac{u}{C_1u_1 - C_2u_2}\right). \end{aligned} \tag{4.22}$$

The following alternative formula for v' may be useful:

$$v' = u\frac{C_1v_2 - C_2v_1}{C_1u_1 - C_2u_2}. \tag{4.23}$$

Formula (4.22) furnishes, by a set of purely algebraic steps, an explicit new solution u', v', q' from three given solutions, i.e. u, v, q and its Bäcklund transformed solutions u_1, v_1, q_1 and u_2, v_2, q_2, obtained by the Darboux transformation (4.6) with parameters C_1 and C_2, respectively.

4.3. Non-localized two-soliton solution

Starting from the trivial solution (4.8) one can construct two one-soliton solutions with different parameters and, then, superimpose them with the above expression (4.22) to get a two-soliton solution. However, this method does not work in the reduced case, since the superposition formula does not preserve the reduction property nor the regularity of the solution. The difficult task is to choose the intermediate solutions u_1, v_1, q_1 and u_2, v_2, q_2 in such a way that the final solution u', v', q' is regular and satisfies the reduction requirement $u' = \overline{v}'$.

It is convenient, first, to rewrite the superposition formula in terms of the eigenfunctions $\chi^{(1)}$ and $\chi^{(2)}$ of the spectral problems (4.4) and (4.5) with potentials $q = 0$, $u = a$, $v = b$ and spectral parameters $h^{(1)}$ and $h^{(2)}$, respectively. We have from (4.6)

$$
\begin{aligned}
q_1 &= -2\partial_s\left(\frac{\chi^{(1)}_{1,s}}{\chi^{(1)}_1}\right) & \qquad q_2 &= -2\partial_s\left(\frac{\chi^{(2)}_{1,s}}{\chi^{(2)}_1}\right)\\
u_1 &= -\frac{a}{C_1}\frac{\chi^{(1)}_{1,s}}{\chi^{(1)}_1} & \qquad u_2 &= -\frac{a}{C_2}\frac{\chi^{(2)}_{1,s}}{\chi^{(2)}_1}\\
v_1 &= -C_1\frac{\chi^{(1)}_2}{\chi^{(1)}_1} & \qquad v_2 &= -C_2\frac{\chi^{(2)}_2}{\chi^{(2)}_1}.
\end{aligned}
\tag{4.24}
$$

Inserting these expressions into the superposition formula (4.22), after some algebra, we obtain

$$
\begin{aligned}
q' &= -2\partial_s\left[\frac{\chi^{(1)}_{1,ss}\chi^{(2)}_1 - \chi^{(2)}_{1,ss}\chi^{(1)}_1}{\chi^{(1)}_{1,s}\chi^{(2)}_1 - \chi^{(2)}_{1,s}\chi^{(1)}_1}\right]\\
u' &= \frac{a}{C_1C_2}\frac{\chi^{(1)}_{1,ss}\chi^{(2)}_{1,s} - \chi^{(2)}_{1,ss}\chi^{(1)}_{1,s}}{\chi^{(1)}_{1,s}\chi^{(2)}_1 - \chi^{(2)}_{1,s}\chi^{(1)}_1}\\
v' &= C_1C_2\frac{\chi^{(1)}_1\chi^{(2)}_2 - \chi^{(2)}_1\chi^{(1)}_2}{\chi^{(1)}_{1,s}\chi^{(2)}_1 - \chi^{(2)}_{1,s}\chi^{(1)}_1}.
\end{aligned}
\tag{4.25}
$$

We choose two couples of real parameters $H^{(1)}$, $K^{(1)}$ ($H^{(1)} \neq K^{(1)}$) and $H^{(2)}$, $K^{(2)}$ ($H^{(2)} \neq K^{(2)}$) such that

$$
{H^{(1)}}^3 + {K^{(1)}}^3 = {H^{(2)}}^3 + {K^{(2)}}^3 \tag{4.26}
$$

and define

$$
\omega^{(j)} = -\tfrac{1}{2}(H^{(j)} + K^{(j)}) \pm \mathrm{i}\tfrac{\sqrt{3}}{2}(H^{(j)} - K^{(j)}) \qquad (j = 1, 2). \tag{4.27}
$$

Then, we consider ($j = 1, 2$)

$$
\begin{aligned}
\chi^{(j)}_1 &= \mathrm{e}^{\mathrm{i}Y^{(j)}}\left[\alpha^{(j)}_1\,\mathrm{e}^{-X^{(j)}} + \alpha^{(j)}_2\,\mathrm{e}^{X^{(j)}}\right]\\
\chi^{(j)}_2 &= -\mathrm{i}b\,\mathrm{e}^{\mathrm{i}Y^{(j)}}\left[\alpha^{(j)}_1\frac{1}{\omega^{(j)}}\,\mathrm{e}^{-X^{(j)}} + \alpha^{(j)}_2\frac{1}{\overline{\omega}^{(j)}}\,\mathrm{e}^{X^{(j)}}\right]
\end{aligned}
\tag{4.28}
$$

with

$$
X^{(j)} = \omega^{(j)}_{\mathrm{Im}}\big(s - 4\omega^{(j)}_{\mathrm{Re}}y\big) \tag{4.29}
$$

$$
Y^{(j)} = \omega^{(j)}_{\mathrm{Re}}s + |\omega^{(j)}|^2 y \tag{4.30}
$$

and $\alpha^{(j)}_i$ arbitrary constants to be chosen. These vector functions $\chi^{(1)}$ and $\chi^{(2)}$ satisfy the spectral problems (4.4) and (4.5) with potentials $q = 0$, $u = a$, $v = b$ and spectral parameters $h^{(1)}$ and $h^{(2)}$, respectively, when ab and $h^{(1)}$ and $h^{(2)}$ are chosen as follows:

$$
ab = {H^{(1)}}^3 + {K^{(1)}}^3 = {H^{(2)}}^3 + {K^{(2)}}^3 \tag{4.31}
$$

$$
{h^{(1)}}^2 = 3H^{(1)}K^{(1)} \qquad {h^{(2)}}^2 = 3H^{(2)}K^{(2)}. \tag{4.32}
$$

Therefore, by inserting the $\chi^{(j)}$ defined in (4.28) into (4.25) we get a new solution of (3.5). We take advantage of the freedom in the choice of the constants $\alpha^{(j)}_i$ in order to get a regular solution.

In fact by taking

$$\alpha_1^{(1)} = \sqrt{(\overline{\omega}^{(1)} - \overline{\omega}^{(2)})(\overline{\omega}^{(1)} - \omega^{(2)})} \qquad \alpha_2^{(1)} = \sqrt{(\omega^{(1)} - \omega^{(2)})(\omega^{(1)} - \overline{\omega}^{(2)})}$$
$$\alpha_1^{(2)} = \sqrt{(\overline{\omega}^{(1)} - \overline{\omega}^{(2)})(\omega^{(1)} - \overline{\omega}^{(2)})} \qquad \alpha_2^{(2)} = \sqrt{(\omega^{(1)} - \omega^{(2)})(\overline{\omega}^{(1)} - \omega^{(2)})}$$

we obtain the regular solution

$$q' = -2\partial_s \frac{M(y,s)}{D(y,s)} \tag{4.33}$$

$$u' = -\frac{a}{2C_1C_2}\frac{N(y,s)}{D(y,s)} \tag{4.34}$$

$$v' = -\frac{bC_1C_2}{2|\omega^{(1)}\omega^{(2)}|^2}\frac{\overline{N(y,s)}}{D(y,s)} \tag{4.35}$$

where

$$\begin{aligned} M(y,s) = {} & |\omega^{(1)} - \omega^{(2)}|\,\mathrm{Im}(\omega^{(1)} + \omega^{(2)})\sinh(X^{(1)} + X^{(2)}) \\ & + |\omega^{(1)} - \overline{\omega}^{(2)}|\,\mathrm{Im}(\omega^{(1)} + \overline{\omega}^{(2)})\sinh(X^{(1)} - X^{(2)}) \end{aligned} \tag{4.36}$$

$$\begin{aligned} N(y,s) = {} & |\omega^{(1)} - \omega^{(2)}|\big(\overline{\omega}^{(1)}\overline{\omega}^{(2)}\,\mathrm{e}^{X^{(1)}+X^{(2)}} + \omega^{(1)}\omega^{(2)}\,\mathrm{e}^{-X^{(1)}-X^{(2)}}\big) \\ & + |\omega^{(1)} - \overline{\omega}^{(2)}|\big(\overline{\omega}^{(1)}\omega^{(2)}\,\mathrm{e}^{X^{(1)}-X^{(2)}} + \omega^{(1)}\overline{\omega}^{(2)}\,\mathrm{e}^{-X^{(1)}+X^{(2)}}\big) \end{aligned} \tag{4.37}$$

$$D(y,s) = |\omega^{(1)} - \omega^{(2)}|\cosh(X^{(1)} + X^{(2)}) + |\omega^{(1)} - \overline{\omega}^{(2)}|\cosh(X^{(1)} - X^{(2)}). \tag{4.38}$$

This solution satisfies the constraint $\overline{u}' = v'$ if

$$|C_1C_2|^2\frac{b}{a} = |\omega^{(1)}\omega^{(2)}|^2 \tag{4.39}$$

or, equivalently, recalling (4.17), if

$$\left|\frac{a}{C_1C_2}\right|^2 = -2\frac{\omega_{\mathrm{Re}}^{(1)}}{|\omega^{(2)}|^2}. \tag{4.40}$$

In conclusion, taking into account that u' and v' are defined up to a constant phase, the two-soliton solution is given by

$$q' = -2\partial_s \frac{M(y,s)}{D(y,s)} \tag{4.41}$$

$$u' = \frac{1}{\sqrt{2}}\frac{\sqrt{-\omega_{\mathrm{Re}}^{(1)}}}{|\omega^{(2)}|}\frac{N(y,s)}{D(y,s)} \tag{4.42}$$

for any choice of the two complex parameters $\omega^{(1)}$ and $\omega^{(2)}$ satisfying the constraints

$$\omega_{\mathrm{Re}}^{(1)}|\omega^{(1)}|^2 = \omega_{\mathrm{Re}}^{(2)}|\omega^{(2)}|^2 \tag{4.43}$$

$$\omega_{\mathrm{Re}}^{(j)} < 0 \quad \omega_{\mathrm{Im}}^{(j)} \neq 0 \qquad (j = 1, 2). \tag{4.44}$$

Note that q' vanishes at large s, while u' goes to a constant.

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Appendix

We study properties of the Jost solution ψ^+ defined by the coupled system of integral equations in (3.9) for potentials q, u and v satisfying (3.11).

Inserting ψ_2^+ from the second equations into the first one of (3.9) we have

$$\psi_1^+ = \frac{1}{k+\mathrm{i}0}U(s,k) + \frac{1}{k+\mathrm{i}0}C(k) + \int_{-\infty}^{s} \mathrm{d}s' \frac{\mathrm{e}^{\mathrm{i}k(s-s')}-1}{2\mathrm{i}k}\left[q\psi_1^+ + \mathrm{i}u\int_{s'}^{+\infty}\mathrm{d}s''v\psi_1^+\right] + \int_{s}^{+\infty} \mathrm{d}s' \frac{\mathrm{e}^{-\mathrm{i}k(s-s')}-1}{2\mathrm{i}k}\left[q\psi_1^+ + \mathrm{i}u\int_{s'}^{+\infty}\mathrm{d}s''v\psi_1^+\right] \tag{A.1}$$

where

$$U(s,k) = -\frac{1}{2}\int_{-\infty}^{s}\mathrm{d}s'\mathrm{e}^{\mathrm{i}k(s-s')}u(s') - \frac{1}{2}\int_{s}^{+\infty}\mathrm{d}s'\,\mathrm{e}^{-\mathrm{i}k(s-s')}u(s') \tag{A.2}$$

$$\begin{aligned} C\left[\psi_1^+\right](k) &= \frac{1}{2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}s'\left[q(s')\psi_1^+(s') + \mathrm{i}u(s')\int_{s'}^{+\infty}\mathrm{d}s''v(s'')\psi_1^+(s'')\right] \\ &= \frac{1}{2\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}s'\left[q(s') + \mathrm{i}v(s')\int_{-\infty}^{s'}\mathrm{d}s''u(s'')\right]\psi_1^+(s',k). \end{aligned} \tag{A.3}$$

Note that for potentials q, u and v satisfying (3.11), $U(s,k)$ is bounded, $|U(s,k)| \leqslant M$, and analytical in upper half plane of k , $C[\psi](k)$ is a linear functional of $\psi(k,s)$ independent of s and bounded in the domain of the k plane where ψ is bounded.

If we introduce Φ and Ψ defined by the following integral equations,

$$\Phi = U(s,k) + \int_{-\infty}^{s}\mathrm{d}s'\frac{\mathrm{e}^{\mathrm{i}k(s-s')}-1}{2\mathrm{i}k}\left[q\Phi + \mathrm{i}u\int_{s'}^{+\infty}\mathrm{d}s''v\Phi\right] + \int_{s}^{+\infty}\mathrm{d}s'\frac{\mathrm{e}^{-\mathrm{i}k(s-s')}-1}{2\mathrm{i}k}\left[q\Phi + \mathrm{i}u\int_{s'}^{+\infty}\mathrm{d}s''v\Phi\right] \tag{A.4}$$

$$\Psi = 1 + \int_{-\infty}^{s}\mathrm{d}s'\frac{\mathrm{e}^{\mathrm{i}k(s-s')}-1}{2\mathrm{i}k}\left[q\Psi + \mathrm{i}u\int_{s'}^{+\infty}\mathrm{d}s''v\Psi\right] + \int_{s}^{+\infty}\mathrm{d}s'\frac{\mathrm{e}^{-\mathrm{i}k(s-s')}-1}{2\mathrm{i}k}\left[q\Psi + \mathrm{i}u\int_{s'}^{+\infty}\mathrm{d}s''v\Psi\right] \tag{A.5}$$

then

$$\psi_1^+ = \frac{1}{(k+\mathrm{i}0)}\left(\Phi + C\left[\psi_1^+\right](k)\Psi\right). \tag{A.6}$$

If we prove that $\Phi(s,k)$ and $\Psi(s,k)$ are bounded and analytical in the upper half plane of k, since

$$C\left[\psi_1^+\right](k) = \frac{C[\Phi](k)}{k - C[\Psi](k)} \tag{A.7}$$

we deduce that $C\left[\psi_1^+\right](k)$ is analytical in the upper half plane with poles at the zeros of $k - C[\Psi](k)$ and bounded, once the poles are subtracted. We conclude from (A.6) that $\psi_1^+(s,k)$ is analytical in the upper half plane of k with a factorizing singular behaviour $1/k+\mathrm{i}0$ at $k = 0$ and possible poles and, once the poles and the origin are subtracted, bounded in s and k.

The only property of $U(s,k)$ that we will use in the following is that $U(s,k)$ is analytical in the upper half plane of k and bounded in k and s. Therefore, it is enough to consider (A.4) since the integral equation (A.5) can be considered a special case with $U(s,k) = 1$, which is trivially bounded and analytical.

Exchanging the order of the integrals in (A.4) we get

$$\Phi = \frac{M}{6}\left(\sum_{j=1}^{3}\Phi_j + \sum_{j=-1}^{-3}\Phi_j\right) \tag{A.8}$$

where the Φ_j are solutions of the integral equations

$$\Phi_j(s,k) = \frac{U(s,k)}{M} + \int \mathrm{d}s' K_j(s,s',k)\Phi_j(s',k) \qquad (j = \pm1, \pm2, \pm3) \tag{A.9}$$

with kernel

$$K_{\pm1}(s,s',k) = \theta(\pm(s-s'))q(s')\frac{\mathrm{e}^{\pm\mathrm{i}k(s-s')}-1}{2\mathrm{i}k} \tag{A.10}$$

$$K_{\pm2}(s,s',k) = \theta(\pm(s-s'))v(s')\int_{\mp\infty}^{s'}\mathrm{d}s''\frac{\mathrm{e}^{\pm\mathrm{i}k(s-s'')}-1}{2k}u(s'') \tag{A.11}$$

$$K_{\pm3}(s,s',k) = \pm\theta(s'-s)v(s')\int_{\mp\infty}^{s}\mathrm{d}s''\frac{\mathrm{e}^{\pm\mathrm{i}k(s-s'')}-1}{2k}u(s''). \tag{A.12}$$

All the equations (A.9) are solved by a series

$$\Phi_j(s,k) = \sum_{n=0}^{+\infty}\Phi_j^{(n)}(s,k) \tag{A.13}$$

constructed through the recursion relation

$$\Phi_j^{(n+1)}(s,k) = U(s,k) + \int \mathrm{d}s' K_j(s,s',k)\Phi_j^{(n)}(s',k) \qquad \Phi_j^{(0)}(s,k) = U(s,k). \tag{A.14}$$

We obtain a majorant $\widehat{\Phi}_j^{(n)}$ of $\left|\Phi_j^{(n)}\right|$ by considering the corresponding term of a series expansion of the solution of the integral equation

$$\widehat{\Phi}_j(s,k) = 1 + \int |\mathrm{d}s'|\widehat{k}_j(s,s',k)\widehat{\Phi}_j(s',k) \tag{A.15}$$

where $\widehat{k}_j(s,s',k)$ is a convenient majorant of $K_j(s,s',k)$, i.e.

$$|K_j(s,s',k)| \leqslant \widehat{k}_j(s,s',k). \tag{A.16}$$

If the series $\sum_n \widehat{\Phi}_j^{(n)}$ is uniformly convergent for $k_{\mathrm{Im}} \geqslant 0$, then each iterated term $\Phi_j^{(n)}$ is absolutely bounded by the nth term of a uniformly convergent series and since the $\Phi_j^{(n)}$ are analytical for $k_{\mathrm{Im}} \geqslant 0$ and continuous as $k_{\mathrm{Im}} \to 0$, we conclude that $\Phi(s,k)$ is an analytical function of k for $k_{\mathrm{Im}} \geqslant 0$, continuous as $k_{\mathrm{Im}} \to 0$ and bounded if $\widehat{\Phi}(s,k)$ is bounded.

We must, therefore, construct the kernels $\widehat{k}_j(s,s',k)$ for each of the above equations (A.15). We follow a procedure analogous to that exposed in [18].

Recalling that

$$\left|\frac{\sin Z}{Z}\right| \leqslant 2\gamma \frac{\mathrm{e}^{|\mathrm{Im} Z|}}{1+|Z|} \tag{A.17}$$

for an appropriate constant γ and that

$$0 \leqslant x - y \leqslant I(-x)I(y) \qquad \text{for} \quad I(x) = 1 + |x|\theta(-x) \tag{A.18}$$

we have for $s - s' \geqslant 0$ and $k_{\mathrm{Im}} \geqslant 0$

$$\left|\frac{\mathrm{e}^{\pm\mathrm{i}k(s-s')} - 1}{2\mathrm{i}k}\right| \leqslant \gamma \begin{cases} \frac{2}{|k|} \\ I(\mp s)I(\pm s'). \end{cases} \tag{A.19}$$

Therefore, the majorant kernels are of the form

$$\widehat{k}_{\pm j}(s, s', k) = \theta(\pm(s - s'))\alpha_{\pm j}(s, k)\beta_{\pm j}(s') \qquad j = 1, 2 \tag{A.20}$$

$$\widehat{k}_{\pm 3}(s, s', k) = \theta(s' - s)\alpha_{\pm 3}(s, k)\beta_{\pm 3}(s') \tag{A.21}$$

with two possible choices of α_j and β_j, i.e.

$$\begin{aligned} &\alpha_{\pm 1}(s, k) = \gamma I(\mp s) && \beta_{\pm 1}(s') = I(\pm s')|q(s')| \\ &\alpha_{\pm 2}(s, k) = \gamma I(\mp s) && \beta_{\pm 2}(s') = \pm|v(s')| \int_{\mp\infty}^{s'} \mathrm{d}t\, I(\pm t)|u(t)| \\ &\alpha_{\pm 3}(s, k) = \pm\gamma I(\mp s) \textstyle\int_{\mp\infty}^{s} \mathrm{d}t\, I(\pm t)|u(t)| && \beta_{\pm 3}(s') = |v(s')| \end{aligned} \tag{A.22}$$

or

$$\begin{aligned} &\alpha_{\pm 1}(s, k) = \frac{2\gamma}{|k|} && \beta_{\pm 1}(s') = |q(s')| \\ &\alpha_{\pm 2}(s, k) = \frac{2\gamma}{|k|} && \beta_{\pm 2}(s') = \pm|v(s')| \textstyle\int_{\mp\infty}^{s'} \mathrm{d}t\, |u(t)| \\ &\alpha_{\pm 3}(s, k) = \pm\frac{2\gamma}{|k|} \int_{\mp\infty}^{s} \mathrm{d}t\, |u(t)| && \beta_{\pm 3}(s') = |v(s')|. \end{aligned} \tag{A.23}$$

Then, the integral equations (A.15) are of the form

$$F_{\pm}(s) = 1 \pm \int_{s}^{\pm\infty} \mathrm{d}s'\alpha(s)\beta(s')F_{\pm}(s'). \tag{A.24}$$

Their solution is given by

$$F_{\pm}(s) = \sum_{n=0}^{+\infty} F_{\pm}^{(n)}(s) \tag{A.25}$$

where

$$F_{\pm}^{(n+1)}(s) = \int_{s}^{\pm\infty} \mathrm{d}s'\alpha(s)\beta(s')F_{\pm}^{(n)}(s') \qquad F_{\pm}^{(0)}(s) = 1. \tag{A.26}$$

By induction one can prove that

$$n!F_{\pm}^{(n+1)}(s) = \alpha(s)\int_{s}^{\pm\infty} \mathrm{d}s'\beta(s')\left(\int_{s}^{s'} \mathrm{d}t\alpha(t)\beta(t)\right)^{n} \tag{A.27}$$

and, therefore,

$$F_{\pm}(s) = 1 \pm \alpha(s)\int_{s}^{\pm\infty} \mathrm{d}s'\beta(s')\exp\left[\pm\int_{s}^{s'} \mathrm{d}t\alpha(t)\beta(t)\right]. \tag{A.28}$$

If

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \alpha(t)\beta(t) < +\infty \qquad \int_{-\infty}^{+\infty} \mathrm{d}t\, \beta(t) < +\infty \tag{A.29}$$

then the $F_{\pm}(s)$ are defined by a uniformly convergent series. In addition, we have

$$F_{\pm}(s) - 1 \leqslant C\alpha(s) \int_{s}^{+\infty} \mathrm{d}s' \beta(s') \qquad \text{with} \quad C = \exp \int_{-\infty}^{+\infty} \mathrm{d}t\, \alpha(t)\beta(t). \tag{A.30}$$

The conditions in (A.29) are satisfied for all integral equations if the potentials q, u and v satisfy (3.11).

Therefore, we conclude that the $\Phi_i(s, k)$ and $\Psi_i(s, k)$ $(i = 1, 2, \ldots, 6)$ are analytical in the upper half plane of k. Moreover, since the majorant integral equations satisfy (A.30) with α and β as in (A.22) and (A.23), the $\Phi_i(s, k)$ and $\Psi_i(s, k)$ are also bounded in s and k.

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