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Nonlinear energy transmission in the gap

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Abstract

Numerical simulations of the scattering of a linear plane wave incoming onto a nonlinear medium (sine–Gordon) reveals that: (i) nonlinearity allows energy transmission in the forbidden band, (ii) this nonlinear transmission occurs beyond an energy threshold of the incoming wave, (iii) the process begins (at the threshold) with large amplitude breathers, and then energy is generically transmitted both by kink–antikink pairs and breathers. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

From the Fermi–Pasta–Ulam recursion phenomenon [1], nonlinear lattice and their numerical simulations have played a key role in the development of nonlinear studies. Since then a whole branch of theoretical physics has evolved with the birth of the concepts of soliton [2] and integrability [3], for continuous, discrete and higher-dimensional systems [4]. These discoveries have had a enormous impact on the understanding of nonlinear physics [5].

One of the fundamental studied problem is the energy localization (and transport) by generation (and propagation) of nonlinear coherent structures (solitons, breathers, nonlinear modes) by many different processes such as external forcing, noise, local potential, inhomogeneous forcing, anharmonic perturbation, *modulated* external forcing, etc. [6]. We note that nowadays this problem is more largely studied in the context of discrete systems where intrinsic nonlinear

modes (breathers for short) have been shown to play the main part in storage and transport of energy [7].

Another basic question is the scattering of waves incoming onto a nonlinear medium. Two classes of problems related to wave scattering have been considered in detail: the *resonant wave coupling* which includes self-induced transparency (SIT) [8], stimulated Raman scattering [9,10], two-photon propagation [11], etc., and the *self-modulation* resulting from Kerr effect in optical media [12,13]. Within these two classes of problems, solitons have been shown to play the central role in propagating energy in regions of parameter (frequency, energy, etc.) usually forbidden by the linear theory.

The resonant wave coupling describes the process of interaction of an electromagnetic radiation with a two-level medium through Brillouin selection rules (the nonlinearity is extrinsic). For SIT the linear theory would conclude with resonant absorption of radiation while nonlinearity allows for perfect transparency. The model equations read as coupled Maxwell–Bloch system and the scattering of waves results in a boundary-value problem for the reduced system. The point there is that this boundary-value problem is integrable —

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see also [14] — and it has been shown that the soliton (actually a kink for the phase) is the means which propagates energy through the medium without loss or distortion, provided the input pulse obeys the *area theorem* [8]. Hence in SIT, the soliton is created by a *sufficiently energetic* boundary condition. In the inverse spectral transform scheme, this boundary value problem actually maps to a Cauchy problem and the area theorem simply states that the initial datum must produce at least one discrete eigenvalue [8,14].

In the case of self-modulation, the nonlinearity finds its origin in the medium itself (intrinsic Kerr effect). A process similar to SIT (corresponding to a Cauchy problem in the IST scheme), but with different origin, occurs in fiber guides in nonlinear regime (and with anomalous dispersion) where a sufficiently energetic input light pulse evolves as a soliton of the nonlinear Schrödinger equation (NLS) along the fiber [12,15]. Here and in SIT, the energy flows with solitons created by a localized high energy pulse.

A different soliton effect arises in the study of the propagation of light in a medium with forbidden bands (photonic band gaps) for which the concept of *gap soliton* have been introduced [16]. There it has been shown that the nonlinearity in a dielectric periodic thin-film stack (medium with spatial periodic variation of the refraction index) allows for transmission of radiation inside the gap by means of an envelope soliton. In short, the presence of an envelope soliton in the medium shifts locally the gap and allows for transparency [17]. The model for the slowly varying envelope is the NLS equation [18]. For that reason (transmission of radiation in the gap), and the related switching properties (transmission with soliton, reflection without), photonic band gap media are the subject of active research, see, e.g., [19]. Notice that the *creation* of the envelope soliton in those media is an open question. Indeed, while the nonlinearity, via NLS, offers possible existence of a soliton, the mechanism of its generation itself is still to be understood. Such a mechanism has been for instance analytically described in the case of stimulated Raman scattering in [10] with the tool of the inverse spectral transform applied to boundary value problems (another example can be found in [20]).

Then the mechanism of soliton creation by radiation is of fundamental interest and we consider here the (nonlinear) scattering of waves from a quite ele-

mentary point of view. Namely, we explore the possibility for a nonlinear medium to build up nonlinear coherent structures under “*exposition*” to input radiation. The problem we are interested in, very simple in linear systems, concerns the reflection and transmission of a plane wave in an inhomogeneous medium with band pass type dispersion relation: a plane wave created in a medium 1 is sent onto a medium 2 with a different dispersion relation. In particular, total reflection occurs when the incoming wave of medium 1 falls onto medium 2 with a frequency inside a forbidden band. We consider then this generic question in the case when the medium 2 is intrinsically nonlinear and modelled by an integrable equation (sine–Gordon).

2. Statement of the problem

We consider then two different media defined by the wave equations

$$x < 0: \quad u_{tt} - c_1^2 u_{xx} + \omega_1^2 u = 0, \quad (1)$$

$$x > 0: \quad u_{tt} - c_2^2 u_{xx} + \omega_2^2 \sin u = 0, \quad (2)$$

with continuity conditions in $x = 0$ (inferred here from the natural discrete version of the above system, see later). The scattering of a wave of given frequency Ω and normalized amplitude incoming from medium 1 to medium 2, when Ω lies in the phonon gap of the nonlinear medium ($\Omega < \omega_2$), is described *in the linear limit* by the general solution (the normalization is chosen such as to have an incident wave $A \sin(Kx - \Omega t)$)

$$x < 0: \quad u(x, t) = \frac{1}{2i} e^{-i\Omega t} [A e^{iKx} + B e^{-iKx}] + \text{c.c.}, \quad (3)$$

$$x > 0: \quad u(x, t) = \frac{1}{2i} e^{-i\Omega t} [C e^{-\kappa x} + D e^{\kappa x}] + \text{c.c.}, \quad (4)$$

with K and κ given from Ω by

$$\Omega^2 = \omega_1^2 + K^2 c_1^2 = \omega_2^2 - \kappa^2 c_2^2. \quad (5)$$

While the linear theory produces an evanescent wave ($D = 0$), the nonlinearity does not prevent in principle nonlinear superposition¹ of exponentials

¹ See the comment at the end of the Letter.

in (4), possibly allowing energy transmission to the nonlinear medium by means of a soliton. We address then the question whether or not solitons build up under scattering and allow nonlinear transmission of radiation within the gap.

The present study is fully based on numerical simulations and, before going further, it is worth stressing that we will discover that indeed solitons (actually kink–antikink pairs and breathers) are created to allow for energy transmission. Moreover, this process is in essence different either from self-induced transparency in two-level media, or else from gap solitons in photonic band gap media, because reductive perturbation method (or multi-scale expansions) do not apply: the incoming radiation creates *large amplitude* nonlinear objects.

3. Method

System (1), (2) is first mapped to the following discrete analog where the first nonlinear pendulum u_1 is coupled to the last linear pendulum u_0 by the spring c_1 of the linear chain:

$$\begin{aligned}
 n < 1: \quad & \ddot{u}_n - c_1^2(u_{n+1} - 2u_n + u_{n-1})/h^2 \\
 & + \omega_1^2 u_n = 0, \\
 n = 1: \quad & \ddot{u}_1 - c_2^2(u_2 - u_1)/h^2 + c_1^2(u_1 - u_0)/h^2 \\
 & + \omega_2^2 \sin u_1 = 0, \\
 n > 1: \quad & \ddot{u}_n - c_2^2(u_{n+1} - 2u_n + u_{n-1})/h^2 \\
 & + \omega_2^2 \sin u_n = 0.
 \end{aligned} \tag{6}$$

This maps to (1), (2) in the continuous limit $h \rightarrow 0$ and has natural unconstrained continuity conditions in $n = 1$.

In order to generate in the linear medium 1 an *incoming plane wave* we follow the work [21] by making the change of variable from $u_n(t)$ to $f_n(t)$:

$$u_n(t) = f_n(t) + AP(n) \sin(Knh - \Omega t), \tag{7}$$

where Ω is given from K by (5), where A is the amplitude of the incoming wave and where the function $P(n)$ (a polynomial) is chosen such as to vary from the value 1 for large negative n to the value 0 far before the nonlinear medium 2. To set things clear, we have used a medium 1 made of a (linear) chain of 10500 oscillators and the polynomial $P(n)$ varies from the

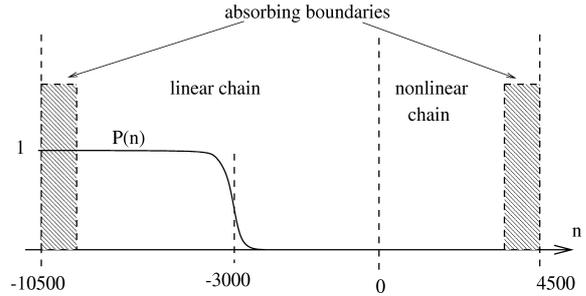


Fig. 1. Grid data for numerical integration.

value 1 at $n = -10500$ to zero around $n = -3000$. The medium 2 (nonlinear chain) is constituted of 4500 pendula. Finally, to simulate the infinite line, strong damping is included at both ends of the interval (so-called *absorbing boundaries*) on a length of about 450 particles. All these settings are summarized in Fig. 1. The link with the continuous system is achieved with step $h = 1/30$, for which we shall check that the generated nonlinear structures are continuous objects.

Expression (7) is inserted in system (6) which is then solved for $f_n(t)$ as a system of $2N$ coupled first-order differential equations in the variable t ($N = 15000$) with the subroutine DOPRI 5. This method allows to launch a plane wave in the system and its efficiency is first checked by equating both media to the linear one and we obtain a nice plane wave propagating without distortion for very long times (longer than the actual nonlinear experiments) as seen in Fig. 2(a) where the parameters of the (homogeneous, linear) medium are $c_1^2 = 8$, $\omega_1^2 = 1$ and the input wave has frequency $\Omega = 1.29$ and amplitude $A = 1.42$. Note, in particular, that the created wave does vanish for large $x = nh$ thus testing the efficiency of the absorbing boundaries (the left-hand side absorbing boundary from $x = -300$ to $x = -315$ is not represented but works as well).

Second, we have verified that, if the right-hand side medium is linear, a wave with frequency in the phonon gap is indeed totally reflected as seen in Fig. 2(b), in which the amplitude of the incident wave largely exceeds the threshold amplitude required for nonlinear transmission described later. Here the parameters are $c_1^2 = 8$, $\omega_1^2 = 1$ for the left (linear) medium and $c_2^2 = 1$, $\omega_2^2 = 3$ for the right (still linear) medium. The

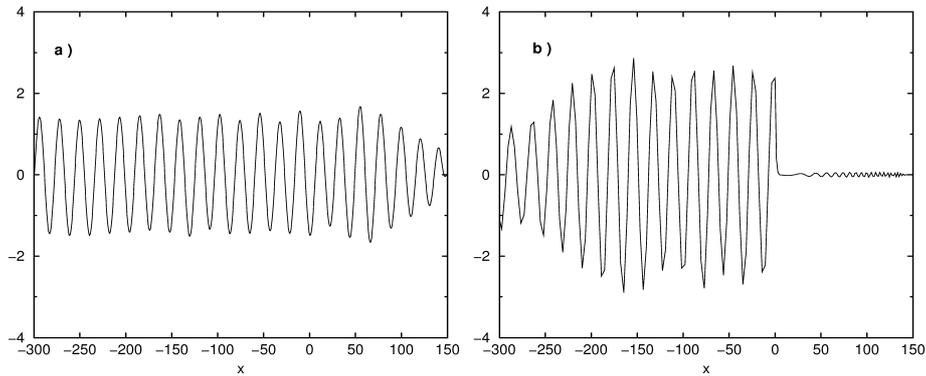


Fig. 2. Graphs of $u(x, t)$ at $t = 250$. (a) Evidence of plane wave generation in a linear homogeneous medium. (b) Evidence of total reflection on the stop gap of a linear right-hand side medium.

frequency of the input plane wave is $\Omega = 1.29 < \omega_2$ and its amplitude $A = 1.5$.

4. Results

Thanks to the scale invariance of system (1), (2), we can always fix the parameters of one of the two media and vary the others. Hereafter we shall then chose the parameters of the nonlinear medium to be

$$c_2^2 = 1, \quad \omega_2^2 = 3, \quad (8)$$

and vary the linear medium (c_1 and ω_1) and the incident plane wave $A \sin(Kx - \Omega t)$. We obtain the following results.

1. An incoming wave with frequency inside the gap of the nonlinear medium ($\Omega < \omega_2$) is transmitted if its average energy E exceeds a threshold energy E_0 . For the choice (8) we found $E_0 = 1.68$. The value of E we refer to is obtained by averaging over one period the Hamiltonian of the linear chain, i.e.,

$$E = \frac{1}{2} A^2 \Omega^2. \quad (9)$$

This energy transmission is accomplished by the generation of kink–antikink pairs and large amplitude breathers. The former occur alone when the incident energy equals the threshold E_0 . Both breathers and kink–antikinks then propagate in the medium. To illustrate this, a typical experiment at energy $E > E_0$ is shown in Fig. 3 where the

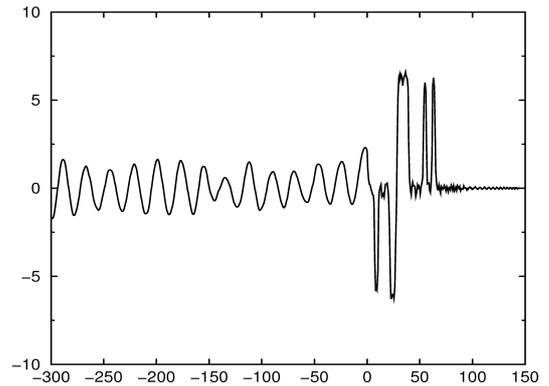


Fig. 3. Solution $u(x, t)$ on the full line at $t = 120$ for a incident wave at $\Omega = 1.29$ and energy $E = 1.8$. The parameters of the linear chain are $c_1^2 = 8$, $\omega_1^2 = 1$.

function $u(x, t)$ is drawn at time $t = 120$ in both media. By looking the picture at different times, we can state that the transmitted wave is made of two breathers followed by two kink–antikink pairs, followed themselves by a new breather. The various behaviors are then depicted in Fig. 4 where the wave in the nonlinear medium is drawn for different times either for an incident energy $E < E_0$ (total reflection), $E = E_0$ (breather generation) and $E > E_0$ (kink–antikink and breather creation). Note that the average full width at half maximum of the breather, e.g., the one appearing in Fig. 4 at $E = E_0$, contains about 73 particles, hence being a continuous object.

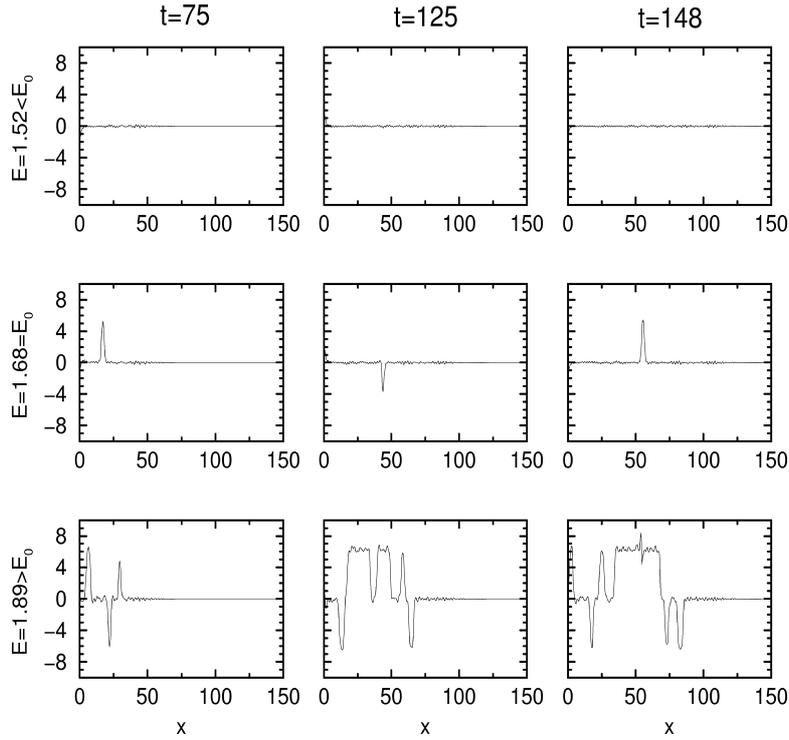


Fig. 4. Representation of nonlinear oscillators $u(x, t)$ for an incident wave at $\Omega = 1.29 < \omega_2$ as a function of x for different times. The energy E is varied with the amplitude A of incident wave. The parameters of the linear chain are $c_1^2 = 8, \omega_1^2 = 1$.

2. The found threshold energy E_0 for nonlinear transmission has the following properties: it does not depend on the incident frequency Ω and it does not depend either on the *nature* of the linear left-hand side medium (electromagnetic vacuum for $\omega_1^2 = 0$ or dielectric medium for $\omega_1^2 \neq 0$). This is illustrated in Fig. 5 where the dots denote values above which nonlinear transmission becomes effective as functions of the incident frequency Ω .
3. This transmission process does not occur if the nonlinear system does not possess soliton solutions. For instance, replacing the nonlinearity $\sin u$ by any of its truncated Taylor expansions, we do not obtain energy transmission. We illustrate this in Fig. 6 where $\sin u$ has been replaced by its Taylor expansion up to order u^7 . This indicates that this process does not result from a shift of the stop gap due to nonlinearity. If such would be the case, one would also observe transmission with the Taylor expansion of $\sin u$ (and, moreover, one would also

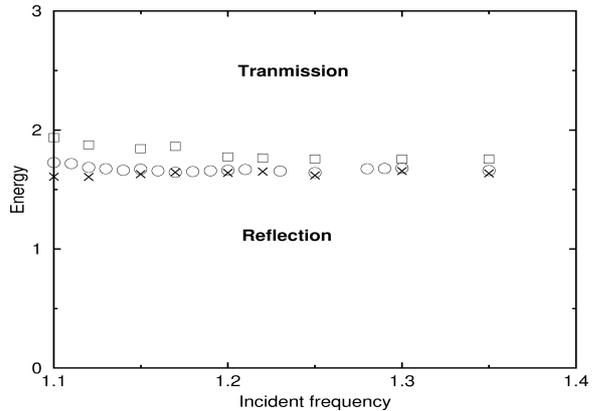


Fig. 5. Energy threshold for nonlinear absorption. The parameters of the linear chain are: (\square) $c_1^2 = 10, \omega_1^2 = 0$, (\circ) $c_1^2 = 8, \omega_1^2 = 1$, (\times) $c_1^2 = 16, \omega_1^2 = 1$.

observe transmission of radiation as *waves*, not as nonlinear objects).

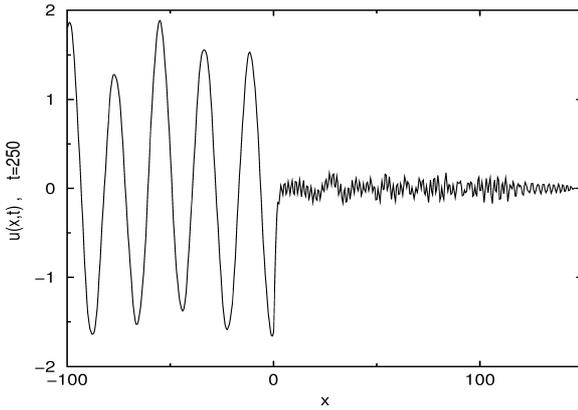


Fig. 6. This figure shows the solution $u(x, t)$ at $t = 250$ in the case $E = 1.89 > E_0 = 1.68$ as in Fig. 4 when $\sin u$ is replaced with its Taylor expansion at order u^7 .

5. Conclusion

By modelling the scattering of a plane wave incoming onto a nonlinear medium, we have discovered the phenomenon that we call *nonlinear transmission*, and which relies on the property of a soliton-bearing system (like sine–Gordon but unlike nonlinear Klein–Gordon) of being able to superimpose nonlinearly exponential waves. With respect to gap solitons where the incident radiation flows through the medium thanks to the presence of a soliton who shifts the gap (or “*opens the door*”), here the radiation is totally converted into localized large amplitude nonlinear objects. These nonlinear objects cannot be described by perturbation approaches but require fully nonlinear treatment (e.g., via inverse spectral transform).

This nonlinear transmission has immediate important implications such as *nonlinear absorption* of radiation (which could play a role in the interpretation of anomalous absorption bands), or else *nonlinear tunnelling*, both in continuous and discrete systems. All this issues will be displayed in forthcoming works but we want to stress here that the theoretical interpretation of the threshold E_0 of incident average energy is still to be understood.

Note that we have checked that this threshold is not due to discreteness effects (Peierls barrier to overcome) by duplicating the numerical simulations with different steps. Varying the step size from $h = 1/10$

to $h = 1/100$ we have obtained the same threshold of nonlinear transmission.

6. Comment

The nonlinear superposition of exponentials, say $\exp[\xi_1(x, t)]$ and $\exp[\xi_2(x, t)]$, is obtained by just writing down the two-soliton solution of sine–Gordon under the form

$$u(x, t) = 4 \arctan \left[\frac{a_1 + a_2}{a_1 - a_2} \frac{e^{\xi_2} - e^{\xi_1}}{1 + e^{\xi_2 + \xi_1}} \right], \quad (10)$$

where a_1 and a_2 are two arbitrary parameters, and where

$$2\xi_j = (a_j + 1/a_j)\omega_2/c_2(x - x_j) + (a_j - 1/a_j)\omega_2(t - t_j), \quad (11)$$

with arbitrary constants x_j and t_j (note that those values allows us to fix arbitrarily amplitudes and phases of both exponentials). It is remarkable that the functions $\exp[\xi_j(x, t)]$ are precisely solutions of the linearized sine–Gordon equation

$$v_{tt} - c_2^2 v_{xx} + \omega_2^2 v = 0, \quad (12)$$

and formula (10) indeed furnishes a *nonlinear superposition recipe*.

The kink–antikink results from parameters a_j real and of *same sign* (opposite signs produce the 2-kink or the 2-antikink solutions). The breather is obtained by assuming complex constants a_j with $a_2 = \bar{a}_1$. A quite simple exercise allows one to have insight on the creation and propagation of such nonlinear structures. In particular the kink and antikink will be seen to separate as time runs, as on the numerical experiments in Fig. 4 (use, for instance, the parameters $a_1 = -0.85$, $a_2 = -0.9$, $x_1 = 5$, $x_2 = 5$, $t_j = 0$ in the normalized units, i.e., for $c_2 = \omega_2 = 1$).

We note finally that the stationary breather furnishes a striking instance of a nonlinear superposition of the evanescent waves in (4). Indeed, defining

$$a_1 = \bar{a}_2 = -\kappa \frac{c_2}{\omega_2} + i \frac{\Omega}{\omega_2}, \quad (13)$$

we obtain for $x_j = t_j = 0$:

$$u(x, t) = 4 \arctan \left[i \frac{\kappa c_2}{\Omega} \frac{e^{-i\Omega t - \kappa x} - e^{i\Omega t - \kappa x}}{1 + e^{-2\kappa x}} \right], \quad (14)$$

where relation (5) between κ and Ω simply means that we have chosen $|a_j| = 1$ (for a *stationary breather*).

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